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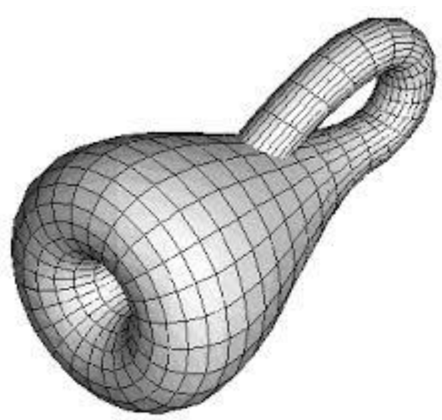
Continuum

The St. Clair County Community College Journal of Mathematics



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The St. Clair County Community College Journal of Mathematics

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This is the first issue of Continuum, a journal we hope to continue to publish on a yearly or twice-yearly basis. We hope that our colleagues, both in the mathematics and science areas and across the college, will enjoy our efforts. We also hope to appeal to students, particularly those who enjoy mathematics. Our intent is to offer articles on a variety of mathematical and pedagogical subjects and at a variety of levels, though all of the contents will be challenging reading.

The reason for the existence of this journal, originally, was simply that some of us on the mathematics faculty had some ideas we wanted to share - ideas that we had used with some success in our classes. From time to time, we make a small discovery or find a new way to present our content. We realized that, although many professional journals exist, there was no ready market for exactly the sort of material we had. When one of us originated the concept of an in-house journal, it quickly became apparent that this would serve several purposes. We could not only communicate our ideas more fully and fluently within our discipline, but we could share them with students and colleagues. We are proud of what we do in the mathematics division, and we have too few opportunities to introduce our colleagues to these ideas.

It is our hope that this journal will stimulate questions and discussion, even criticism. To this end, we invite letters containing reactions and suggestions, as well as questions and corrections. We will publish a sample of these letters in our next issue. Please direct them to the editors.

It is also our hope that other disciplines will emulate us and share their exciting discoveries in a similar way. We look forward to some competition!

Continuum includes brief summaries of some extraordinary student work as well. The names of the students are listed in the article.

The Derivative of a Radical Function

Nick Goins

To compute the derivative of the square root function using the limit definition of the derivative, we need to multiply by the conjugate. For a different radical function, what algebraic steps are required in order to compute the derivative? More precisely, what expression do we have to multiply by so that we can simplify the difference quotient in such a way that the factor of h in the denominator will cancel? The answer lies in the factorization formulas for the difference of powers. Consider the following factorization formulas,

$$\boxed{\begin{aligned} a^2 - b^2 &= (a-b)(a+b) \\ a^3 - b^3 &= (a-b)(a^2+ab+b^2) \\ a^4 - b^4 &= (a-b)(a+b)(a^2+b^2) \end{aligned}}$$

For example, given the expression $\sqrt{x+h} - \sqrt{x}$ if we make the identification

$$a \ \leftrightarrow \ \sqrt{x+h} \qquad\qquad b \ \leftrightarrow \ \sqrt{x}$$

then the formula for the difference of two squares gives

$$\begin{aligned} \left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right) &= \left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2 \\ &= (x+h) - x \\ &= h \end{aligned}$$

That is, multiplying the expression $\sqrt{x+h} - \sqrt{x}$ by its conjugate, will eliminate the radicals. What then would we multiply $\sqrt[3]{x+h} - \sqrt[3]{x}$ by in order to eliminate the radicals? The hope is to find an expression $C_3(x)$ so that

$$\begin{aligned} \left(\sqrt[3]{x+h} - \sqrt[3]{x}\right)C_3(x) &= \left(\sqrt[3]{x+h}\right)^3 - \left(\sqrt[3]{x}\right)^3 \\ &= (x+h) - x \\ &= h \end{aligned}$$

The expression $C_3(x)$ is found using the factorization formulas above.

Example: Let $C_3(x) = \left(\left(\sqrt[3]{x+h}\right)^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right)$. Multiply $\sqrt[3]{x+h} - \sqrt[3]{x}$ by $C_3(x)$.

Solution:

$$\left(\sqrt[3]{x+h} - \sqrt[3]{x}\right)\left(\left(\sqrt[3]{x+h}\right)^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right)$$

$$= \sqrt[3]{x+h}\left(\sqrt[3]{x+h}\right)^2 + \sqrt[3]{x+h}\,\sqrt[3]{x+h}\sqrt[3]{x} + \sqrt[3]{x+h}\left(\sqrt[3]{x}\right)^2 - \sqrt[3]{x}\left(\sqrt[3]{x+h}\right)^2 - \sqrt[3]{x}\sqrt[3]{x+h}\sqrt[3]{x} - \sqrt[3]{x}\left(\sqrt[3]{x}\right)^2$$

$$= \left(\sqrt[3]{x+h}\right)^3 - \left(\sqrt[3]{x}\right)^3 = (x+h) - x = h$$

Thus, if we were computing the derivative of the cube root function by simplifying the difference quotient, we would compute the following

$$\frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} = \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \cdot \frac{\left(\sqrt[3]{x+h}\right)^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2}{\left(\sqrt[3]{x+h}\right)^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2}$$

$$= \cdots = \frac{(x+h) - h}{h\left(\left(\sqrt[3]{x+h}\right)^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right)} = \frac{1}{\left(\sqrt[3]{x+h}\right)^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2}$$

Definition: The **conjugate** of the expression $\sqrt[n]{x+h} - \sqrt[n]{x}$ is an expression $C_n(x)$, which is a factor used to eliminate the radicals.

Example: Use the limit of a difference quotient definition of the derivative to compute the derivative of $f(x) = \sqrt[3]{x}$.

Solution: The derivative is

$$\frac{d}{dx}\left[\sqrt[3]{x}\right] = \lim_{h \to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h}$$

To compute the limit, we would need to eliminate the h from the denominator, and this is done by rationalizing the numerator. This algebra was done in the previous example, so we will use that work below

$$\lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}=\lim_{h\to 0}\frac{1}{\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}=\frac{1}{\left(\sqrt[3]{x+0}\right)^2+\sqrt[3]{x+0}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}$$

$$=\frac{1}{3\left(\sqrt[3]{x}\right)^2}$$

□

Note: The conjugate for a radical expression, $\sqrt[n]{x+h}-\sqrt[n]{x}$, is found using the factorization formula for the difference of n^{th} powers. That is, the factorization formula for a^n-b^n is given by

$$a^n-b^n=(a-b)C_n$$

Example: To find the conjugate of $\sqrt[4]{x+h}-\sqrt[4]{x}$, we will use the factorization formula

$$a^4-b^4=(a-b)(a+b)(a^2+b^2),$$

Identifying $\sqrt[4]{x+h}$ with a and $\sqrt[4]{x}$ with b and substituting these expressions into $(a+b)(a^2+b^2)$, we obtain the conjugate. See exercise $1f$.

Example: To find the factorization formula for a^8-b^8, we will use the difference of squares three time

$$a^8-b^8=(a^4)^2-(b^4)^2=(a^4-b^4)(a^4+b^4)=(a^2-b^2)(a^2+b^2)(a^4+b^4)$$

$$=(a-b)(a+b)(a^2+b^2)(a^4+b^4)$$

Similarly we can find the factorization formula for $a^{16}-b^{16}$

$$a^{16}-b^{16}=(a-b)(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$$

Notice that we can write these factorization formulas in the following way

$$a^8 - b^8 = (a-b)(a^{2^0} + b^{2^0})\left(a^{2^1} + b^{2^1}\right)\left(a^{2^2} + b^{2^2}\right)$$

and

$$a^{16} - b^{16} = (a-b)(a^{2^0} + b^{2^0})\left(a^{2^1} + b^{2^1}\right)\left(a^{2^2} + b^{2^2}\right)\left(a^{2^3} + b^{2^3}\right)$$

This looks messier, but it allows us to find similar factorization formulas (and hence, conjugates) quickly, using the pattern. For instance the factorization of $a^{32} - b^{32}$ is

$$a^{32} - b^{32} = (a-b)(a^{2^0} + b^{2^0})\left(a^{2^1} + b^{2^1}\right)\left(a^{2^2} + b^{2^2}\right)\left(a^{2^3} + b^{2^3}\right)\left(a^{2^4} + b^{2^4}\right)$$

The following theorem states the formula for the conjugate when the index of the radical is a power of 2. The above example provides motivation for the formula.

Theorem: The factorization formula for $a^{2^n} - b^{2^n}$ is given by

$$\boxed{a^{2^n} - b^{2^n} = (a-b)\left(a^{2^0} + b^{2^0}\right)\left(a^{2^1} + b^{2^1}\right)\left(a^{2^2} + b^{2^2}\right)\left(a^{2^3} + b^{2^3}\right)\ldots\left(a^{2^{n-1}} + b^{2^{n-1}}\right)}$$

The conjugate is therefore given by

$$\boxed{C_{2^n} = \left(a^{2^0} + b^{2^0}\right)\left(a^{2^1} + b^{2^1}\right)\left(a^{2^2} + b^{2^2}\right)\left(a^{2^3} + b^{2^3}\right)\ldots\left(a^{2^{n-1}} + b^{2^{n-1}}\right)}$$

Note: Using product notation, we can write the conjugate of $\sqrt[2^n]{x+h} - \sqrt[2^n]{x}$ as

$$C_{2^n} = \prod_{k=1}^{n-1}\left[\left(\sqrt[2^n]{x+h}\right)^{2^k} + \left(\sqrt[2^n]{x}\right)^{2^k}\right]$$

More generally to find a factorization formula for $x^n - c^n$, we can divide this expression by $x - c$.

Exercises

1. List the conjugate of the expression

$a)\ \sqrt{x+2} - \sqrt{2}$

$b)\ \sqrt{x^2+3} - \sqrt{3}$

$c)\ \sqrt[3]{x+4} - \sqrt[3]{x}$

$d)\ \sqrt[3]{x+5} - \sqrt[3]{5}$

$e)\ \sqrt[4]{x+4} - \sqrt[4]{x}$

$f)\ \sqrt[4]{x+h} - \sqrt[4]{h}$

$g)\ \sqrt[8]{x+4} - \sqrt[8]{x}$

2. Multiply the expression and its conjugate for each exercise in problem 1.

3. Use the limit of a difference quotient definition of the derivative to compute the derivative of the function

$a)\ f(x) = \sqrt{x+5}$ $\qquad b)\ f(x) = \sqrt[3]{x+1}$ $\qquad c)\ f(x) = \sqrt[4]{x}$ $\qquad d)\ f(x) = \sqrt[8]{x}$

4. Find the factorization formula for $a^6 - b^6$. You will need to use the difference of squares, the difference of cubes and the sum of cubes. Then, state the conjugate for $\sqrt[6]{x+h} - \sqrt[6]{x}$.

5. Find the factorization formula for $a^9 - b^9$. Then, state the conjugate for $\sqrt[9]{x+h} - \sqrt[9]{x}$.

6. Use synthetic division to divide $f(x)$ by $d(x)$, where c is a fixed constant

$a)\ f(x) = x^2 - c^2, \qquad d(x) = x - c$

$b)\ f(x) = x^3 - c^3, \qquad d(x) = x - c$

$c)\ f(x) = x^4 - c^4, \qquad d(x) = x - c$

$d)\ f(x) = x^5 - c^5, \qquad d(x) = x - c$

$e)\ f(x) = x^6 - c^6, \qquad d(x) = x - c$

$f)\ f(x) = x^7 - c^7, \qquad d(x) = x - c$

7. Consider the factorization formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. We can rewrite this as

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$$

Thus, we can find the factorization formulas, and hence a conjugate, using division. Use division to find the factorization of $a^{11} - b^{11}$ and then state the conjugate of $\sqrt[11]{x+h} - \sqrt[11]{x}$

8. Use the conjugate from exercise 4 to compute the derivative of $f(x) = \sqrt[6]{x}$.

MATH: We Need to Talk

Brian Robertson

Imagine being a student in a collegiate mathematics classroom. You probably picture a sterile environment that is in no way, shape, or form conducive to conversation. It is possible to spend the entire semester sitting in the back row, head down, without speaking a word. Sure you can listen to your instructor's lecture, take notes, and probably pass most assessments. You may even end up doing well in the course; but will you leave with any meaningful mathematical understanding? At some point, as scary as it may seem for students and instructors alike, students need to talk with each other to gain a genuine understanding of mathematical concepts.

Research tells us that complex knowledge and skills are learned through social interaction (Vygotsky 1978; Lave and Wenger 1991). Furthermore:

> Social interaction provides us with the opportunity to use others as resources, to share our ideas with others, and to participate in the joint construction of knowledge. In the mathematics classroom, high quality discussions support student learning of mathematics by helping students learn how to communicate their ideas, making students' thinking public so it can be guided in mathematically sound directions, and encouraging students to evaluate their own and each other's mathematical ideas (Smith and Stein, 2011).

It i

The Erdos number is an excellent example of the collaboration amongst mathematicians. Named after the famed prolific late mathematician Paul Erdos, this number calculates a person's degree of separation from Mr. Erdos. A direct collaborator with Mr. Erdos would have an Erdos number of one. A subsequent collaborator would have an Erdos number of two and so on. An estimated 260,000 mathematicians have an Erdos number between one and 13 and nearly 90% of all mathematicians today have an Erdos number of eight or less. Furthermore, "b**y 2000, less than half of all mathematics papers were by a single author, about a third were by two authors, about an eighth by three authors, and 3% by four or more authors ("Facts about Erdos Number", 2015)."** Clearly mathematicians talk a lot about

mathematics with colleagues! To encourage this type of meaningful collaboration in the post-secondary mathematics classroom, students need exposure to high-level instructional tasks.

Such tasks are created in three phases. First, the instructor introduces students to a mathematical problem that incorporates important mathematical ideas, tools available for solving it, and what students are expected to produce. Then, students collaborate on accomplishing the task. Finally, students share their findings.

An example of such a task is an error-analysis assignment. For this activity students are asked to examine a fictional student's incorrect answer. Students should be encouraged to use mathematical reasoning to not only justify why the given answer is incorrect but to also produce a correct solution. This type of activity will stimulate conversations about common mathematical misconceptions including how they occur and how to avoid them. Furthermore students have the opportunity to use each other for resources, to communicate their thinking, and to evaluate mathematical ideas.

For students to partake in authentic mathematical learning they need to engage socially with classmates. Students need the opportunity to examine their own mathematical thinking and that of others. It is through this examination and conversation the full potential of mathematics is realized. Let's give students the chance to behave like "real" mathematicians and share their mathematical ideas!

References

Erdos Number Project. "*Facts About Erdos Numbers.*" Oakland University. 29 July 2015. Web. 04 April 2014. https://wwwp.oakland.edu/enp/trivia/

Lave, J., and E. Wenger, *Situated Learning: Legitimate Peripheral Participation.* N.Y.: Cambridge University Press, 1991.

Smith, M., and M Stein, *5 Practices for Orchestrating Mathematics Discussions.* V.A.: Reston.: The National Council Teachers of Mathematics, Inc., 2011.

Vygotsky, L.S. *Mind in Society: The Development of Higher Psychological Processes.* Cambridge, Mass.: Harvard University Press, 1978.

Linear Transformations: Derivatives and Integrals

Jeffrey VanHamlin

The derivative of polynomials of degree 2 or less:

Consider the polynomials of degree 2 or less which is the vector space P_2. The general vector p in P_2 is

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

The standard basis of P_2 is

$$S = \{1, x, x^2\} = \{1 + 0x + 0x^2, 0 + x + 0x^2, 0 + 0x + x^2\}$$

Since D_x is a linear transformation we know that if $S = \{1, x, x^2\}$ is a basis in P_2 then

$$D_x(S) = \{D_x(1), D_x(x), D_x(x^2)\}$$

is a basis in $D_x(P_2)$.

Writing this linear combination in Matrix form gives

$$p(x) = a_0 + a_1 x + a_2 x^2 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

and

$$S = \{1, x, x^2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Every vector in P_2 can be written as a linear combination of its basis. For example

$$p(x) = 2 - 5x + 3x^2 = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

In order to find the derivative linear transformation on $p(x)$ we must determine the derivative on each of the basis. This can be verified by the following

$$D_x(P_2) = \frac{d}{dx}(a_0 + a_1x + a_2x^2)$$

$$= \frac{d}{dx}(a_0) + \frac{d}{dx}(a_1x) + \frac{d}{dx}(a_2x^2)$$

$$= a_0\frac{d}{dx}(1) + a_1\frac{d}{dx}(x) + a_2\frac{d}{dx}(x^2)$$

$$= a_0\frac{d}{dx}\begin{bmatrix}1\\0\\0\end{bmatrix} + a_1\frac{d}{dx}\begin{bmatrix}0\\1\\0\end{bmatrix} + a_2\frac{d}{dx}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= a_0D_x\begin{bmatrix}1\\0\\0\end{bmatrix} + a_1D_x\begin{bmatrix}0\\1\\0\end{bmatrix} + a_2D_x\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Since the derivative of a second degree polynomial is a first degree polynomial. The transformation will be $T: P_2 \longmapsto P_1$. The derivative on each of the basis are

$$D_x\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{d}{dx}(1 + 0x + 0x^2) = 0 + 0x = \begin{bmatrix}0\\0\end{bmatrix}$$

$$D_x\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{d}{dx}(0 + 1x + 0x^2) = 0 + 0x = \begin{bmatrix}1\\0\end{bmatrix}$$

$$D_x\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{d}{dx}(0 + 0x + 1x^2) = 0 + 2x = \begin{bmatrix}0\\2\end{bmatrix}$$

Resulting in the linear combination of

$$D_x(P_2) = \frac{d}{dx}(a_0 + a_1x + a_2x^2)$$

$$= a_0D_x\begin{bmatrix}1\\0\\0\end{bmatrix} + a_1D_x\begin{bmatrix}0\\1\\0\end{bmatrix} + a_2D_x\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= a_0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Therefore

$$D_x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Giving the matrix for finding the derivative of a second degree polynomial.

$$\frac{d}{dx}(a_0 + a_1 x + a_2 x^2) = D_x \cdot p(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

or

$$T(p) = D_x p = Ap = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Example 1: Determine the derivative of the polynomial by using matrix multiplication

a. $p(x) = 2 - 5x + 3x^2$

$$\frac{d}{dx}(2 - 5x + 3x^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix} = -5 + 6x$$

b. $p(x) = -4 + 5x - 2x^2$

The linear transformation $T(p) = D_x p = Ap$ is a matrix of size 2×3. This verifies that

$$T\colon P_2 \longmapsto P_1$$

transforms a vector space of dimension 3 to a vector space of dimension 2.

Integration of polynomials of degree 2 or less:

Define $T: P_1 \longmapsto P_2$ be a linear transformation by integration on the general polynomial of degree 2 or less into a polynomial with degree 3 or less. The size of the derivative matrix is 2×3 because T is a linear transformation on the vector space P_1 of dimension 2 onto the vector space P_2 of dimension 3.

$$T(p) = I_x p = \int p(x)dx$$

In this section I will be disregarding the constant C to force the integral to be a linear transformation.

Consider the polynomials of degree 1 or less which is the vector space P_1. The general vector p in P_1 is

$$p(x) = a_0 + a_1 x.$$

The basis on P_1 is

$$S = \{1, x\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

Therefore the basis on $I_x(P_1)$ is

$$I_x(S) = \left\{\mathrm{I}_x \begin{bmatrix}1\\0\end{bmatrix}, I_x \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

In order to find the integral linear transformation on $p(x)$ we must determine the integral on each of the basis.

This can be verified by the following

$$I_x(P_2) = \int (a_0 + a_1 x)dx$$

$$= \int (a_0)dx + \int (a_1 x)\, dx$$

$$= a_0 \int (1)dx + a_1 \int (x)\, dx$$

$$= a_0\, I_x \begin{bmatrix}1\\0\end{bmatrix} + a_1\, I_x \begin{bmatrix}0\\1\end{bmatrix}$$

Since the integral of a second degree polynomial is a third degree polynomial. The transformation will be

$T: P_1 \longmapsto P_2$. The integral on each of the basis are

$$I_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \int dx = x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$I_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \int x \, dx = \frac{x^2}{2} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$

Therefore

$$I_x(P_2) = \int (a_0 + a_1 x) dx$$

$$= a_0 \, I_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \, I_x \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= a_0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Therefore the integration matrix is

$$I_x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

The matrix for finding the integral of a first degree polynomial.

$$T(p) = I_x p = Ap = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Example 2: Determine the integral of the polynomial using matrix multiplication.

a. $p(x) = 3 - 4x$

$$\int (3-4x)dx = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} = 3x - 2x^2$$

b. $p(x) = -5 + 3x$

Note: We are missing the constant C. This could be introduced through matrix addition. So the correction to

integration would be

$$\int (a_0 + a_1x)dx = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} C \\ 0 \\ 0 \end{bmatrix} = C + a_0x + \frac{a_1}{2}x^2$$

Derivative of polynomials of degree 1 or less:

Determining the derivative from P_1 to P_0 can be done with an identical method. The basis for P_1 is

$$S = \{1, x\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$D_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{d}{dx}(1 + 0x) = 0 = [0]$$

$$D_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{d}{dx}(0 + 1x) = 1 = [1]$$

Note that the matrix for the derivative will be one row less, this is due to the fact the polynomial is of degree zero and will never have a coefficient for x.

Therefore

$$D_x(P_1) = \frac{d}{dx}(a_0 + a_1 x) = a_0 D_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 D_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_0[0] + a_1[1] = [0 \quad 1] \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Giving the matrix for finding the derivative of a first degree polynomial.

$$\frac{d}{dx}(a_0 + a_1 x) = [0 \quad 1] \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

or

$$T(p) = D_x p = Ap = [0 \quad 1] \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Example 3: Determine the derivative of the first degree polynomial by matrix multiplication

a. $p(x) = 5 + 7x$

$$\frac{d}{dx}(5 + 7x) = [0 \quad 1] \begin{bmatrix} 5 \\ 7 \end{bmatrix} = [7] = 7$$

b. $p(x) = 2 - 3x$

The second derivative from P_2 to P_0:

The second derivative of a polynomial is a composition of derivatives.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

If we denote the derivative linear transformation that takes the derivative of a second degree polynomial as

$$T_{2,1}: P_2 \longmapsto P_1 \quad \text{by} \quad D_{2,1}(P_2) = P_1$$

and denote the derivative linear transformation that takes the derivative of a first degree polynomial as

$$T_{1,0}: P_1 \longmapsto P_0 \quad \text{by} \quad D_{1,0}(P_1) = P_0$$

Then the second derivative linear transformation would be

$$T(P_2) = T_{1,0}(T_{2,1}(P_2))$$

Since

$$T_{2,1}(p) = A_{2,1}p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

and

$$T_{1,0}(p) = A_{1,0}p = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Then

$$T(P_2) = T_{1,0}\left(T_{2,1}(P_2)\right)$$

$$= A_{1,0} \cdot \left(A_{2,1} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \right) = \left(A_{1,0} \cdot A_{2,1}\right) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Therefore the second derivative matrix D^2 is

$$D^2{}_{2,0} = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}$$

Taking the second derivative of a degree 2 or less polynomial becomes

$$\frac{d^2}{dx^2}(a_0 + a_1x + a_2x^2) = D^2{}_{2,0}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Example 4: Determine the second derivative of the second degree polynomial.

a. $p(x) = 7 + 4x^2$

$$\frac{d^2}{dx^2}(7 + 0x + 4x^2) = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \\ 4 \end{bmatrix} = [8] = 8$$

b. $p(x) = 5 + x - 3x^2$

Example 5: Determine the first derivative linear transformation matrix from P_3 to P_2.

a. Determine the derivative on the basis of P_3.
b. Adjoin the four resultant 3 x 1 column vectors to find $D_{3,2}$.
c. Verify the derivative of $p(x) = 2 - 5x + x^2 + 2x^3$ would be correct through matrix multiplication.

Example 6: Determine the integral linear transformation from P_2 to P_3.

a. Determine the integral on the basis of P_2.
b. Adjoin the three resultant 4 x 1 column vectors to find $I_{2,3}$.
c. Verify the integral of $p(x) = 3 - x + 4x^2$ would be correct through matrix multiplication.

Example 7: Determine the second derivative matrix of the vector space P_3 to the vector space P_1.

a. Multiply $D_{2,1}$ and $D_{3,2}$ to find $D^2{}_{3,1}$

b. Verify the second derivative matrix is correct by find the second derivative of $p(x) = 2 - 3x + 5x^2 + x^3$ through matrix multiplication.

The Fundamental Theorem of Calculus:

Given a continuous function $f(t)$ on a closed interval from $[a, t]$, then

$$\frac{d}{dx}\int_a^t f(t)dt = f(x)$$

Loosely this says that the derivative of the integral is the function or the integral is similar to the inverse of the derivative.

Using the derivative matrix D_x of the polynomials of degree 2 or less and the integral matrix I_x of polynomial of degree 1 or less verify that the derivative of the integral is the original function.

Given

$$D_x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } I_x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Then

$$\frac{d}{dx}\int_a^t f(t)dt = D_x(I_x(P_2)) = (D_x \cdot I_x)\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

So

$$(D_x \cdot I_x)\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}\right)\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Example 8: Verify the multiplication of the derivative matrix $D_{3,2}$ and the integral matrix $I_{2,3}$ is the 3×3 identity matrix.

The Kernel of the transformation:

The kernel of the transformation is the set of all vectors in the Domain vector space that are transformed to the zero vector in the Range vector space.

$$T: \ker(V) \mapsto \mathbf{0}$$

Since the transformation can be described by matrix multiplication the kernel is the solution set to the linear equation

$$A\boldsymbol{x} = \mathbf{0}$$

Example 9: Determine the kernel of $D_{2,1}$ or $Ker(D_{2.1})$**.** This example is asking, for what second degree polynomials is the derivative equal to zero?

The kernel of $D_{2,1}$ is the solution set of $A_{2,1}p = \mathbf{0}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

written in augmented form gives

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Row reduced echelon form of the matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Note the first three columns are coefficients of a_0, a_1, and a_2. So the matrix represents the values of these coefficients, giving $a_1 = 0$ and $a_2 = 0$. Note there is no restriction of the value of a_0 meaning the value of a_0 is free to be any value.

Therefore the $Ker(D_{2.1}) = a_0 + 0x + 0x^2 = a_0$.

The value of the kernel is appropriate because it says the derivative of a constant is zero.

Example 10: Determine $Ker(D_{3,2})$.

Example 11: Determine $Ker(D_{3,1})$.

Other Vector Spaces:

Linear Combinations of the sine and cosine functions:

Consider the linear combination of $f(x) =$ a $\sin x + b\ cos\ x$. The basis of this linear combination is

$$S = \{\sin x, \cos x\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

The derivative matrix would be

$$D_x\begin{bmatrix}1\\0\end{bmatrix} = \frac{d}{dx}(1\sin x + 0\cos x) = \cos x = \begin{bmatrix}0\\1\end{bmatrix}$$

$$D_x\begin{bmatrix}0\\1\end{bmatrix} = \frac{d}{dx}(0\sin x + 1\cos x) = -\sin x = \begin{bmatrix}-1\\0\end{bmatrix}$$

Therefore

$$D_x = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

Example 12: Determine the derivative of the trigonometric equation using matrix multiplication.

a. $f(x) = 2\sin x - 3\cos x$

$$\frac{d}{dx}(2\sin x - 3\cos x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\sin x + 2\cos x$$

b. $f(x) = 5\sin x + 2\cos x$

Since the derivative matrix D_x is a square matrix we can find the inverse matrix I_x.

$$I_x = {D_x}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Example 12: Determine the integral of the polynomial through matrix multiplication.

a. $f(x) = 2\sin x + 5\cos x$

$$\frac{d}{dx}(2\sin x + 5\cos x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 5\sin x - 2\cos x$$

b. $f(x) = 5\sin x - 3\cos x$

The second derivative:

Since the derivative matrix is a linear transformation that transforms the vector space generated by

$S = \{\sin x, \cos x\}$ onto the <u>same</u> vector space we take the second derivative by using the same matrix.

$$\frac{d^2}{dx^2}(\mathrm{a}\sin x + b\ cos\ x) = D_x \cdot \left(D_x \begin{bmatrix} a \\ b \end{bmatrix}\right) = (D_x \cdot D_x)\begin{bmatrix} a \\ b \end{bmatrix} = (D_x)^2\begin{bmatrix} a \\ b \end{bmatrix}$$

In this case due to the fact the derivative of the vector space is the same vector space.

$$D^2{}_x = (D_x)^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example 13: Determine the second derivative of the functions.

a. $f(x) = 4\sin x - 3\cos x$

$$\frac{d^2}{dx^2}(4\sin x - 3\, cos\, x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3\sin x + 4\cos x$$

b. $f(x) = -5\sin x + 2\cos x$

Example 14: Verify that if we take the fourth derivative of the linear combination

$$f(x) = \mathrm{a}\sin x + b\ cos\ x$$

we obtain the same function.

Non-Standard Basis:

The derivative vector space is generated by determining the derivative of the basis of a vector space. Since the derivative vector space is the same space we can consider the derivative of the general basis a non-standard basis.

Standard basis:

$$S = \{\sin x, \cos x\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

Non-standard basis:

$$D_x(S) = \left\{\frac{d}{dx}(\sin x), \frac{d}{dx}(\cos x)\right\} = \{\cos x, -\sin x\} = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\}$$

It can be checked that both sets S and $D(S)$ span the space and both sets are linearly independent. Therefore both sets are a basis of the vector space.

Note: This only works for certain vector spaces. For this result, derivative transformation must transform the vector space onto itself.

Identity Transformation: If the transformation on a vector space is the same vector space, then the transformation is called an identity transformation.

Theorem: If the transformation is an identity transformation, then the kernel of the transformation is only the zero vector.

$$\text{If } T: V \mapsto V \quad \text{then } \; Ker(T) = \mathbf{0}$$

So if the derivative of a linear combination of sines and cosines is also a linear combination of sines and cosines then the transformation is an identity transformation. So

$$Ker(D_x) = \mathbf{0}$$

Example 15: Verify $Ker(D_x) = \mathbf{0}$.

$$Ker(D_x) = \mathbf{0} \;\; \Leftrightarrow \;\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the augmented matrix form of the linear equation gives

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Giving the solution set of $a = 0$ and $b = 0$. So the kernel of the derivative of a the vector space generated by a linear combinations of sine and cosine is

$$Ker(D_x) = 0 \sin x + 0 \cos x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

Note: The derivative transformation on P_2 is not an identity transformation, because

$$Ker(P_2) = a_0 \neq \mathbf{0}.$$

Therefore the derivative of the basis of P_2 is not a non-standard basis.

Natural Base exponential functions:

Consider the linear combination $f(x) = ae^x + be^{-x}$. The basis of the linear combination is

$$S = \{e^x, e^{-x}\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

Example 16: Determine both the derivative matrix D_x and integral matrix I_x of the linear combination function $f(x) = ae^x + be^{-x}$.

Example 17: Determine both the derivative and the integral of the function $f(x) = 2e^x + 5e^{-x}$ through matrix multiplication.

Example 18: Verify that kernel of the derivative transformation is the zero vector.

Linear combination of functions:

Consider taking the derivative of functions build through the addition of all previous functions. Since the derivative of the sum is the sum of the derivatives and the integral of the sum is the sum of the integrals we can find the derivative or integral of these combination functions.

Example 19: Determine the derivative of the function .

a. $f(x) = 3 + 2x - x^2 - 2\sin x - 7\cos x$

$$\frac{d}{dx}(3 + 2x - x^2 - 2\sin x - 7\cos x)$$

Separate the known vector spaces to use the derivative matrix for each space.

$$= \frac{d}{dx}(3 + 2x - x^2) + \frac{d}{dx}(-2\sin x - 7\cos x)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

Even though these matrices are of the same size, we would not add them. Remember the first matrix represents a polynomial of degree 1 or less and the second represents a linear combination of sines and cosines. Therefore

the resulting derivative becomes

$$= 2 - 2x + 7\sin x - 2\cos x$$

b. $5 - 2x + 3x^2 + \sin x + 4\cos x$

c. $1 - 4x^2 - 2x^3 + 3\sin x - 7e^x + 3e^{-x}$

(hint: do not forget to include the missing terms of each vector space)

Example 20: Use a similar method above to determine the integral of the function.

$$f(x) = 1 + 2x - \sin x + 5\cos x$$

Linear Combination of Products of functions:

Consider the vector space of linear combinations of sine and cosine and the linear combinations of exponential functions.

$$S_{trig} = \{\sin x, \cos x\} \text{ and } S_{exp} = \{e^x, e^{-x}\}$$

If we multiply the basis, the resulting basis and linear combination function would become.

$$S = \{e^x \sin x, e^x \cos x, e^{-x} \sin x, e^{-x} \cos x\}$$

and

$$f(x) = ae^x \sin x + be^x \cos x + ce^{-x} \sin x + de^{-x} \cos x.$$

Note: This cannot be done with any two vector spaces. Since each basis is a continuous function we can multiply these functions and their products are also continuous functions. For example, we cannot multiply the basis of R^2 and R^3.

The basis of the linear combination in matrix form is

$$S = \{e^x \sin x, e^x \cos x, e^{-x} \sin x, e^{-x} \cos x\} = \left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

The derivative

$$D_x \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

$$D_x \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{d}{dx}(e^x \cos x) = e^x \cos x - e^x \sin x = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$$

$$D_x \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \frac{d}{dx}(e^{-x} \sin x) = -e^{-x} \sin x + e^{-x} \cos x = \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$$

$$D_x \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \frac{d}{dx}(e^{-x} \cos x) = -e^{-x} \cos x - e^{-x} \sin x = \begin{bmatrix}0\\0\\-1\\-1\end{bmatrix}$$

Giving a derivative matrix

$$D_x = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Since this is square matrix, we can determine the integral matrix by finding the inverse of the derivative matrix.

$$I_x = {D_x}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0.5 \\ 0 & 0 & -0.5 & -0.5 \end{bmatrix}$$

Example 21: Determine both the derivative and the integral of the following function

$$f(x) = 2e^x \sin x - 7e^x \cos x + 5e^{-x} \sin x + 3e^{-x} \cos x$$

Note: Taking the derivative without using matrix multiplication can be done quite quickly using the product rule four times, but the integral is extremely cumbersome. Each of the four basis functions would require

integration by parts twice to find a repeated integral term. Then combine the integral terms and divide by 2 to determine the integral.

Example 22: Determine the basis of the product of the basis of vector space P_2 and the basis vector space generated by the linear combination of sines and cosines.

Alternative Definitions as a Learning Game

Paul Bedard

It is difficult to calculate the probability of a coincidence. There are too many factors involved, and it is human nature to pick and choose among the many events that happen in any given interval and discover serendipity. This occurred for me this semester, when two events involving alternative definitions happened on the same day, and became the genesis of an interesting assignment. First, I will describe the events.

The first event was a discovery I made in the precalculus book written by my friend and co-editor Nick Goins. I noticed that he defined *even functions* as 'functions whose graphs are symmetric to the y – axis.' I was surprised by this, as it was not the definition I was expecting. Please note: even functions *are* absolutely functions whose graphs are symmetric to the y – axis. I knew that perfectly well. So why, you may be wondering, was I surprised? To understand this, I must digress a bit, and say a few words about the nature of definitions in general and their role in mathematics in particular.

A good definition must do two things. It must include every example of the category being defined, and it must exclude everything else. So, for instance, "animals with four legs" is a poor definition for "dog," since it is far too inclusive. But selecting the specific genetic code of the Jack Russell terrier would be too exclusive.

Definitions in science differ in a fundamental way from those in mathematics. Science concerns itself with actual objects in the world, and a definition in science can be judged by how accurately it describes its object. If the definition is incorrect, the object still exists and it doesn't, of course, change to conform to the mistaken definition.

Mathematical objects, however, do not exist. Or at least, they exist only when we call them into existence by defining them. And if our definition doesn't really properly describe the object we envision, nonetheless it describes what it describes, and that becomes the object being defined. For example, suppose I set out to write a definition for "square." I define a square as "a plane figure with four equal sides." My definition is too inclusive; it includes rhombuses (plane figures with four equal sides but not necessarily four equal angles.) Well, the objects I have defined *do* exist, so my definition points to something, and if I decide to use the word "square" where most other mathematicians would use "rhombus," I am free to do so. In fact,

many mathematical essays include such idiosyncratic definitions.

So, Nick's definition was not wrong; it was simply not typical. The usual definition of even functions is as follows: A function is even if, for every x in the domain of the function, $f(-x) = f(x)$. Let me share an example, to explain this. $f(x) = 3x^2$ is an even function, since

$f(-x) = 3(-x)^2 = 3x^2 = f(x)$

Perhaps the reader is now thinking, "That doesn't seem to have anything to do with symmetry!" However, it is easy to show that symmetry to the y – axis is a consequence of this definition. It is also easy to show that this definition is a consequence of symmetry to the y – axis. Therefore, either statement could serve as the definition, leaving the other to be a fact about even functions which is a consequence of the definition.

The second event, the one that made this a coincidence, was the fact that on that very day I was teaching logarithms to my calculus class. Now, we pull rather a fast one on students when we do this. In algebra classes, we tell students that logarithms are essentially exponents. So, for instance, $\log_3 81 = 4$, because $3^4 = 81$. Familiarize yourself with this by trying a few simple exercises (answers at the end of the article).

a) $\log_3 9$

b) $\log_4 16$

c) $\log_2 16$

d) $\log_7 7^{12}$

Note also that logarithms have certain interesting and most useful properties. For instance, the log of a product is the sum of logarithms: $\log ab = \log a + \log b$. (I have deleted the base, because this rule works with any base, so long as the base is the same in each logarithm.) This property is an immediate consequence of defining logarithms as exponents.

After explaining this much to students, we introduce a noninteger, in fact an irrational, base for logarithms: the number *e*. *e* (called Euler's number since the great 18th century Swiss mathematician Leonhard Euler used it to great effect and probably coined the use of the letter e for this number) is an *irrational number*, a number which cannot be expressed as a fraction, or ratio of integers. Therefore the decimal representation for *e* is endless.

Try this exercise:

e) Show that any number whose decimal representation terminates (comes to an end) can be expressed as a ratio of integers. (This is easy.)

Euler's number is approximately equal to 2.71828, which can be remembered by an old mnemonic device, "By omnibus I traveled to Brooklyn." (I enjoy telling my students that I may be the only person in the world who uses this device backwards: I use my knowledge of the digits of *e* to remind myself how many *l*'s there are in the word *traveled.*)

Students are surprised to learn that we are interested in such a peculiar choice for base. In fact, it makes it very difficult to evaluate logarithms! Unless the number we are taking the logarithm of is itself a power of *e*, then without a calculator or computer it is impossible to do better than a crude approximation. Why, then, are we so insistent upon this choice? In fact, why do we refer to it as the *natural* choice, so that we call logarithms with this base *natural logarithms*? To answer this question, we usually tell students, "You'll find out when you take calculus." We may refer to certain application problems like continuously compounded interest. (An aside: does any reader know of any financial institution that actually compounds interest continuously? Please let the editors know!)

In fact, natural logarithms are THE logarithms as far as mathematicians are concerned. I tell my students that, although a base of ten is "common" in practical usage (and base ten logs are in consequence referred to as *common logarithms*) this convention was only adopted (in all likelihood) because we have ten fingers! I invite my class to speculate with me: if intelligent aliens exist who have 8 fingers, or 14, or none, what would they use as the base of their logarithms? Whatever else they might use, probably 8 or 14 in the above cases, they would surely also use *e*. Natural logarithms get their own function name in American textbooks; we say $\ln x$ rather than $\log_e x$. In fact, we ought to accustom students to recognizing that $\log x$ in mathematical writing invariable denotes the natural, not the common, logarithm.

The reason for this insistence is because logarithms, to a mathematician, are not just useful because they represent exponents. They are useful because

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

In fact, this is not just a useful property for calculus, this is in fact *how we define logarithms* for calculus students. Note the huge shift in values here: whereas *e* seemed a peculiar base in an otherwise orderly, integer-laden function, now the function is *defined* with *e* in mind! Indeed, integer-base logs must now be

justified somehow on the basis of this definition! It is possible, though difficult perhaps for the reader (try it!) to show that the exponent definition of logarithms is a consequence of this one. We can extend to any base, and this definition preserves all the properties we had before, such as $\log ab = \log a + \log b$.

With all this in mind, and Nick's "backward" definition of even functions in mind as well, I conceived of an interesting exercise for students. I would ask them to pick any mathematical object, and write their own definition for it, using some property or consequence of the standard definition. Perhaps their definition would be "bad" in the sense of being either too inclusive, or too exclusive. I would ask them to try to recognize this. I would also ask them to try to prove the standard definition, as a consequence of their new definition.

As an additional example, I suggested the following:

Circle (standard definition): the set of all points in the plane equidistant from a given point (called the center.)

Circle (new definition): any ellipse whose foci coincide (are the same point.)

To see why this makes sense, let us define an ellipse as "the set of points in the plane the sum of whose distances from two fixed points (the foci) is constant."

(An exercise for the reader

f) Prove that the original definition of circle is a consequence of the new definition.)

What is the point of such an assignment? Should I not stick with the standard definitions? Emphatically not! Mathematics is many things: an art, a science, a discipline for the mind. But it is also a game – the most enjoyable and intellectually stimulating game there is. Every mathematician plays in the sandbox by toying with new definitions, making things up, discarding useless or unworthy frivolities like a god creating and destroying worlds. I want my students to see that mathematics is not drudgery and boredom, nor is it rigid and unyielding, but that it involves creativity and freedom. Also, what better way is there to emphasize the importance of definitions and proofs. Karl Weierstrass said, "A mathematician who is not also something of a poet will never be a complete mathematician."

The student results were in most cases satisfactory and in some cases very encouraging. The best was submitted by Brittany Middleton. Brittany decided to redefine the function $f(x) = e^x$, as "a function with derivative equal to the original function." This is a property that e^x has, and is indeed the reason e is

such a natural base for logarithms! (This new definition is a bit too inclusive; for instance, $g(x) = 3e^x$ is also equal to its own derivative. But nonetheless, this is a great property to use to try to define e^x.)

Based on this definition, Brittany wanted to prove the alternate definition,

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

She proceeds to show this as follows:

$$\frac{d}{dx}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\frac{d}{dx}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} n\left(1+\frac{x}{n}\right)^{n-1} * \frac{1}{n} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1}$$

$$= \frac{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

(Note that we have not verified that interchanging the limit and differential operators in this case is valid; however, given the continuity and convergence properties, it can be shown to be valid.)

James Gilbert redefined a square as "two right triangles whose hypotenuses coincide." He hoped to use this definition to prove the standard definition of square: a four-sided plane figure with equal sides and four right angles. If he had said "equilateral right triangles," this would work better. But note how the thought process was very fruitful here – thinking about squares leads to thinking about triangles leads to thinking about equilateral triangles and the Pythagorean theorem….

Ethan Albany redefined a perfect square as "the sum of odd integers 1 + 3 + 5 + …+ *n*" This is a good definition! The standard definition is a number which is equal to $n \cdot n$, where n is an integer. Ethan

used many examples to provide support for the identity of these definitions. For instance:

$$1 = 1$$

$$1 + 3 = 4$$

$$1 + 3 + 5 = 9$$

$$1 + 3 + 5 + 7 = 16$$

Examples don't constitute proof in mathematics, but this proof would require a technique called mathematical induction. I will leave that for some future article.

Janice Fritz defined a very specific object: a particular parabola. She used the standard definition of parabola to define $y = 2 - (x^2/8)$ as "the set of points equidistant from the point $(0, 0)$ and the line $y = 4$." This is a specific instance of the definition of parabolas, but what a good idea to make it specific. Her alternate definition was "a second degree polynomial with vertex at $(0, 2)$ and roots at ± 2."

Can we define a parabola this way? This definition is sufficiently inclusive – it surely includes $y = 2 - (x^2/8)$ – but does it exclude all other parabolas? It sure does! Janice provides a lovely proof, by demonstrating in detail that the focus and directrix of a parabola defined as in the alternate definition could only be those given in the original definition. I leave the details as an exercise to the (patient) reader.

Some students provided definitions that didn't meet the criteria of inclusivity or exclusivity – but they were interesting, instructive, and creative.

Dan Caluya was intrigued by the old mathematical chestnut "There is no way to square the circle." This often misinterpreted problem was posed by the Greeks of antiquity, who wanted to fine a rational value for the area of a square which would be equal to the area of a corresponding circle. Since the area of a circle is πr^2, the area of the corresponding square would be πr^2 as well, so the side length of this square would be $\sqrt{\pi r^2}$, or $r\sqrt{\pi}$.

The famous unsolvability of "squaring the circle" does not suggest that $\sqrt{\pi}$ does not exist. It is all about the fact that $\sqrt{\pi}$ is not a rational number. The Greeks were a little obsessed with important numbers being rational. And with good reason, but that again is another topic.

Dan felt that the unsolvability of squaring the circle might provide a key to a good definition of π. He essentially tried to define π as any number which will create an unsolvable problem of this type. Unfortunately, as Dan himself readily saw, this definition is far too inclusive. Any irrational number has an irrational square root. (Can you prove this?)

An exercise:

g) Can the sum of irrational numbers be rational? The product? Don't overthink!

Alex Scholz was even more ambitious. His new definition has given me more food for thought than any other. He selected exponentiation. He was intrigued by the unusual fact that any nonzero number raised to the power of zero is equal to 1. Alex wanted to define exponentiation solely in terms of this property! This is exciting, but much in the way that cliff diving is exciting – the tumble through space is exhilarating, but eventually you strike the water, and hopefully not the rocks beneath! There are some rocks waiting under the water here. This definition would be far too inclusive. Many functions possess the property $f(0) = 1$. And yet….I sense a glimmer of something. Perhaps if we looked at the *way* in which the

exponentiation leads us to this….but this, again, is a topic for another day.

Pimvipha Rhodes tried to define decimals in a way that emphasized the difference between rational and irrational numbers. Kurtis Rhein wanted to define triangles in terms of trigonometric functions, instead of the other way around. This is also ambitious, and fascinating. Roger Malcolm tried to redefine L'Hopital's Rule, which can be applied multiple times, in terms on the number of times it can be applied!

For me, this assignment and its results were very exciting. I love to push my students into new areas and try to encourage them to think in new ways about old matters. I love the creativity and boldness I saw in these results, even if some rigor was lacking. I feel that a creative mind can be taught rigor far more easily than a rigid mind can be taught creativity.

Answers to exercises

a) 2 b) 2 c) 4 d) 12

e) The positions after the decimal point are tenths, hundredths, thousandths, etc. So, a terminating decimal can always be expressed as a fraction with the digits in the decimal as the numerator, and the denominator as a power of 10. For instance, 0.1234 is $\frac{1234}{10{,}000}$

f) If the foci of an ellipse coincide, then the sum of the distances from any point to the foci is just twice a constant, which is therefore a constant.

g) $\pi + (-\pi) = 0$ which is rational. $\pi\left(\frac{1}{\pi}\right) = 1$ which is rational.

Roth IRA Account: A Recursive Calculator Program

Barry Trippett

$$0 \rightarrow P\ (enter)$$

$$5500 + P \rightarrow P : P * e^{(0.04*1)} \rightarrow P$$

(Press *enter* repeatedly for the number of years that you invest new money.)

Use your calculator's store feature and iterative feature to answer the following questions about Roth IRA retirement accounts.

1) Your current age: ________.

How many years do you have until retirement at age 67?

2) You have set up a Roth IRA savings account January 1st of this year. The account pays 4% annual interest and the interest is compounded continuously $A = Pe^{rt}$. You always add the maximum annual contribution of $5,500 each January 1st. How much would you have in your retirement account in:

5 years:

10 years:

30 years:

When you retire at age 67:

How much of your own money will you have put into the account when you retire?

How much interest did you earn?

3) If you had started your Roth IRA account when you were 13 years old, how much would you have at the age of 67?

How much of your own money will you have put into the account when you retire?

How much interest did you earn?

4) If you had started your Roth IRA account when you were 13 years old and then stopped making contributions at age 45 but you let the amount in the account continue to grow with interest until you retire at age 67, how much money would you have at retirement?

How much of your own money will you have put into the account when you retire?

How much interest did you earn?

5) Any thoughts on the future of your finances?

“And check your work”

Cindie Wade

One of the things I try to encourage my students to do is to check their work. I can still remember Miss Glover, my fifth grade teacher, “forcing” us into checking our work and it was only double digit multiplication, long division, decimal operations and fractions. She would come around the class and if you didn’t have the “check” on the paper next to the problem, she would pull your ear and clearly say, “Check your work”. It never made sense to me then, I thought I was a pretty good math student and darn confident of my answer. However, now I get it!!

I try to convince the students to be sure their answers are reasonable and check them to see if they are correct. There are multiple ways that they can check them, even if it is just to look in the back of the book to see if their odd answers are correct. (I never assign the odds, as it is my hopes that they try them before actually doing the ones that I do assign to be turned in).   There are many hidden math concepts in checking your work and you have to be a little clever to use your calculator. They paid big bucks for the thing, they might as well use it, and not just to get answers and not show any work. I show them how to check with their calculator, using the STO button to simply check their answer to an equation. Or there is the TEST button to check solutions. Yes, I am the one that shows them the PlySmlt App (I call it the PolySmittal), to check their answers to a system of equations or solutions to higher order equations. I show them how to do problems in various ways, not just algebraically, but graph it, see if it works. They can also mentally check their work and I try to encourage them to do “mental math”, whenever possible.

We have all seen answers that just don’t make sense: Sons older than their fathers, the sum of the angles in a triangle that don’t add up to 180°, answers to absolute value equations that give a negative value, extraneous roots that students claim are solutions.

When I taught Math Methods for K-8 students I would very often use a deck of cards that I had made with problems that were done wrong and the students had to figure out what the student had done wrong to get the answer. As a student, I was sometimes left perplexed with not knowing where I made a mistake, so as a teacher, I like to let them know where and what their error was, I leave them a little love note. HOWEVER, if they check their work there would be a less likely chance that they would make those errors.

George Polya (1887-1985) has been my hero for confirming "check your work". George was living when I was in college. I remember hearing about him and his book, "How to Solve It". Yes, I have a copy of it, no, he didn't sign it. The book is actually quite boring and he, himself summed it up by saying there are 4 steps to solving any problem:

1. Understand the problem-You can't solve a problem unless you understand what you are being asked to do. Sometimes you need to read the problem more than once before starting to solve it. Think about what your answer will look like.
2. Devise a plan-There are many ways to solve problems. How are you going to attack it? What method are you going to use? Which one will seem most logical.
3. Carry out the plan-Do what you think would be a good plan. Sometimes you may have "chosen poorly" and you need to restart with a different approach.
4. Look back and **CHECK YOUR WORK**-First see if your answer is reasonable. Does it fit the conditions of the problem? Second, have you answered all of the questions asked of the problem? Third, can you solve it a different way and come up with the same answer?

I think these 4 steps can be applied to even the non-application problems that we assign in math. George's picture hangs in my classroom to remind my students that they can determine

Once in a while I will have students show me that they did, indeed, check their work. They put a paper and pencil check next to their work, or show me what they did on their calculator. I'm impressed and I hope they continue to check their solutions.

Yes, I still collect their assignments and ***check*** them. I need to know what they do or don't know. I am "The Queen of Homework".

An Introduction to Differentiation of Non-Integer Order

Nick Goins

The differential operators discussed in the first three semesters of the calculus sequence are all of integer order. However, in this article I will introduce the basics of fractional order derivatives (or more precisely, differential operators of order α, where α is any real number). One analogy we can make is with exponents. When you first learned about exponents, it was explained that the exponent was a shorthand for how many factors there were. For example, 2^3, was a shorthand for 2 multiplied by itself 3 times. Then when rational exponents were introduced, you tried to understand what it meant to have 2 multiplied by itself one-half of a time (i.e., the square root). Or, what does it mean to multiply 4 by itself, two-thirds of a time. Of course, these ideas of multiplying a number by itself a fraction of a time, was meaningless. What we did was reinterpret the notation. That is, 4 to the one-half power, was the number which you would square to get 4. Similarly, a differential operator of order n (for a positive integer n) describes how many times we differentiate. The following is analogous, in that you know how to differentiate once, twice, etc. How can we differentiate one-half of a time? Just like with square roots, for fractional derivatives, we will have to reinterpret what we mean by these operations.

We will find the fractional derivative of the most common functions encountered in a calculus course, the power functions, polynomials, exponential functions and the sine and cosine functions. The theme will be to first find a general formula for the integer order derivatives, then generalize the formula to allow non-integer order derivatives. Also, there will be numerous examples and theorems provided.

Fractional Derivatives of Power Functions and Polynomials

Consider the power function, $f(x) = x^k$ for an integer k and its first few derivatives

$$f'(x) = kx^{k-1} \qquad f''(x) = k(k-1)x^{k-2} \qquad f'''(x) = k(k-1)(k-2)x^{k-3}$$

Following with this pattern, and for $n \leq k$, we have

$$\frac{d^n}{dx^n}[x^k] = \frac{k!}{(k-n)!}x^{k-n}$$

Thus, in order to generalize the differential operator to fractional orders, and to generalize the above formula to non-integer degrees of the power function, we must generalize the notion of a factorial.

Definition: The **factorial function** is given by $f(n) = n!$, where n is any non-negative integer.

We will generalize this function by finding a continuous extension of it. More precisely, we will find a continuous extension of the shifted factorial function $f(n) = (n-1)!$. The important idea is that we need to extend the concept of a factorial to the non-integer real numbers. The continuous extension we will choose is what is known as the gamma function. We will use Wolfram Alpha extensively throughout what follows.

Definition: The **gamma function** is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$$

For now, all we need to know about this function is given in the next list. If you want to evaluate these using wolfram alpha, you would compute $\Gamma(5.2)$ for example by entering: gamma(5.2) (you should get 32.5781 ...).

1. The gamma function is a continuous extension of the shifted factorial function $f(n) = (n-1)!$.

2. $\Gamma(x+1) = x\Gamma(x)$

3. $\Gamma(n) = (n-1)!$ for every integer n

4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

5. $\Gamma(2) = 1$

Notes:

1. Using the second and fifth properties listed above, we can verify the fourth property. For example

$$\Gamma(5) = \Gamma(4+1) = 4\Gamma(4) = 4\Gamma(3+1) = 4\cdot 3\Gamma(3) = 4\cdot 3\Gamma(2+1) = 4\cdot 3\cdot 2\Gamma(2) = 4!$$

2. In the special case above for the n^{th} derivative of the power function of integer degree k (again with $n \leq k$), we can write the expression for the derivative using the gamma function as follows,

$$\frac{d^n}{dx^n}[x^k] = \frac{\Gamma(k+1)}{\Gamma(k-n+1)}x^{k-n}$$

since

$$\frac{\Gamma(k+1)}{\Gamma(k-n+1)} = \frac{k!}{(k-n)!}$$

Thus, we will use this formula to define the differential operator with real order α,

Theorem: For a real number $\alpha \geq 0$ and for any non-zero real number k, the derivative of the power function $f(x) = x^k$ of order α is

$$\boxed{\frac{d^\alpha}{dx^\alpha}[x^k] = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}}$$

Example: Compute the one-half derivative of $f(x) = x^2$.

Solution:

$$\frac{d^{1/2}}{dx^{1/2}}[x^2] = \frac{\Gamma(2+1)}{\Gamma\left(2-\frac{1}{2}+1\right)}x^{2-\frac{1}{2}} = \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}x^{3/2} = \frac{2!}{\Gamma\left(\frac{5}{2}\right)}x^{3/2}$$

To compute $\Gamma\left(\frac{5}{2}\right)$, we will use the properties listed above. That is

$$\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\Gamma\left(\frac{3}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{4}\sqrt{\pi}$$

Therefore,

$$\frac{d^{1/2}}{dx^{1/2}}[x^2]=\frac{2!}{\Gamma\left(\frac{5}{2}\right)}x^{3/2}=\frac{2!}{\left(\frac{15}{4}\sqrt{\pi}\right)}x^{3/2}=\frac{8}{15\sqrt{\pi}}x^{3/2}$$

□

Example: Compute the one-half derivative of $f(x)=\frac{8}{15\sqrt{\pi}}x^{3/2}$.

Solution:

$$\frac{d^{1/2}}{dx^{1/2}}\left[\frac{8}{15\sqrt{\pi}}x^{3/2}\right]=\frac{8}{15\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\left[x^{3/2}\right]=\frac{8}{15\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{2}+1\right)}x^{\frac{3}{2}-\frac{1}{2}}$$

$$=\frac{8}{15\sqrt{\pi}}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(2)}x^{\frac{2}{2}}=\frac{8}{15\sqrt{\pi}}\frac{\left(\frac{15}{4}\sqrt{\pi}\right)}{1}x=2x$$

□

Thus, combining the two examples above, we have the following result,

$$\frac{d^{1/2}}{dx^{1/2}}\left[\frac{d^{1/2}}{dx^{1/2}}[x^2]\right]=2x=\frac{d}{dx}[x^2]$$

This illustrates the following theorem

Theorem: For non-negative real numbers α and β

$$\boxed{\frac{d^{\alpha}}{dx^{\alpha}}\left[\frac{d^{\beta}}{dx^{\beta}}[f(x)]\right] = \frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}[f(x)]}$$

In the case of integer order derivatives, this result is already known to calculus students. That is, the derivative of a second derivative is the third derivative, for instance.

Example: Using decimal approximations (computed with wolfram alpha), find the one-fifth derivative of $f(x) = x^3$, then find the four-fifths derivative of the result to verify that you would get the first derivative.

Solution: The one-fifth derivative is,

$$\frac{d^{1/5}}{dx^{1/5}}[x^3] = \frac{\Gamma(3+1)}{\Gamma\left(3-\frac{1}{5}+1\right)}x^{3-\frac{1}{5}} = \frac{\Gamma(4)}{\Gamma\left(\frac{19}{5}\right)}x^{3-\frac{1}{5}} \approx \frac{6}{4.6942}x^{14/5} \approx 1.2782x^{14/5}$$

The four-fifths derivative of the one-fifth derivative is

$$\frac{d^{4/5}}{dx^{4/5}}\left[\frac{d^{1/5}}{dx^{1/5}}[x^3]\right] = \frac{d^{4/5}}{dx^{4/5}}\left[1.2782x^{14/5}\right] = 1.2782\frac{d^{4/5}}{dx^{4/5}}\left[x^{14/5}\right]$$

$$= 1.2782\frac{\Gamma\left(\frac{14}{5}+1\right)}{\Gamma\left(\frac{14}{5}-\frac{4}{5}+1\right)}x^{\frac{14}{5}-\frac{4}{5}} \approx 1.2782 \cdot \frac{4.6942}{2}x^{10/5} \approx 3.00006x^2$$

□

Example: Without using decimal approximations, find the four-thirds derivative of $f(x) = 2x^5$, then find the two-thirds derivative of the result to verify that you would get the second derivative.

Solution: The four-thirds derivative is

$$\frac{d^{4/3}}{dx^{4/3}}[2x^5] = 2 \cdot \frac{\Gamma(5+1)}{\Gamma\left(5-\frac{4}{3}+1\right)} x^{5-\frac{4}{3}} = 2 \cdot \frac{\Gamma(6)}{\Gamma\left(\frac{14}{3}\right)} x^{11/3}$$

The two-thirds derivative of the four-thirds derivative is

$$\frac{d^{2/3}}{dx^{2/3}}\left[\frac{d^{4/3}}{dx^{4/3}}[2x^5]\right] = \frac{d^{2/3}}{dx^{2/3}}\left[2 \cdot \frac{\Gamma(6)}{\Gamma\left(\frac{14}{3}\right)} x^{11/3}\right] = 2 \cdot \frac{\Gamma(6)}{\Gamma\left(\frac{14}{3}\right)} \frac{d^{2/3}}{dx^{2/3}}\left[x^{11/3}\right]$$

$$= 2 \cdot \frac{\Gamma(6)}{\Gamma\left(\frac{14}{3}\right)} \cdot \frac{\Gamma\left(\frac{11}{3}+1\right)}{\Gamma\left(\frac{11}{3}-\frac{2}{3}+1\right)} x^{11/3-2/3} = 2 \cdot \frac{\Gamma(6)}{\Gamma\left(\frac{14}{3}\right)} \cdot \frac{\Gamma\left(\frac{14}{3}\right)}{\Gamma(4)} x^{9/3} = 2 \cdot \frac{\Gamma(6)}{\Gamma(4)} x^3 = 2 \cdot \frac{5!}{3!} x^3 = 2 \cdot 5 \cdot 4x^3$$
$$= 40x^3$$

□

Theorem: For $\alpha \geq 0$ and for differentiable functions $f(x)$ and $g(x)$

$$\boxed{\frac{d^\alpha}{dx^\alpha}[f(x) + g(x)] = \frac{d^\alpha}{dx^\alpha}[f(x)] + \frac{d^\alpha}{dx^\alpha}[g(x)]}$$

Note: The above theorem justifies the term-by-term fractional differentiation for polynomials. It also shows that the sum rule for integer order derivatives extends to the non-integer order case. The same is not true for the product and quotient rule.

Example:

$$\frac{d^{4/3}}{dx^{4/3}}[2x^4+3x^3+7x-5]=\frac{d^{4/3}}{dx^{4/3}}[2x^4]+\frac{d^{4/3}}{dx^{4/3}}[3x^3]+\frac{d^{4/3}}{dx^{4/3}}[7x]+\frac{d^{4/3}}{dx^{4/3}}[5]=\cdots$$

Note: A fractional derivative of a constant function may not be 0.

Example: Compute the fractional derivatives of $f(x)=1$.

Solution:

$$\frac{d^\alpha}{dx^\alpha}[1]=\frac{d^\alpha}{dx^\alpha}[x^0]=\frac{\Gamma(0+1)}{\Gamma(0-\alpha+1)}x^{0-\alpha}=\frac{\Gamma(1)}{\Gamma(1-\alpha)}x^{-\alpha}=\frac{1}{\Gamma(1-\alpha)x^\alpha}$$

That is,

$$\boxed{\frac{d^\alpha}{dx^\alpha}[1]=\frac{1}{\Gamma(1-\alpha)x^\alpha}}$$

□

Example: Use the above formula to compute the one-half derivative of $f(x)=1$.

Solution: With $\alpha=\frac{1}{2}$, the above formula gives

$$\frac{d^{1/2}}{dx^{1/2}}[1]=\frac{1}{\Gamma\left(1-\frac{1}{2}\right)x^{1/2}}=\frac{1}{\Gamma\left(\frac{1}{2}\right)\sqrt{x}}=\frac{1}{\sqrt{\pi}\sqrt{x}}$$

Then,

$$\frac{d^{1/2}}{dx^{1/2}}\left[\frac{d^{1/2}}{dx^{1/2}}[1]\right]=\frac{d^{1/2}}{dx^{1/2}}\left[\frac{1}{\sqrt{\pi}\sqrt{x}}\right]=\frac{d^{1/2}}{dx^{1/2}}\left[\frac{1}{\sqrt{\pi}}x^{-1/2}\right]=\frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(-\frac{1}{2}-\frac{1}{2}+1\right)}x^{-\frac{1}{2}-\frac{1}{2}}=\frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(0)}x^{-1}$$

This expression is actually undefined, since the gamma function has a vertical asymptote at the input 0. Thus, in order to compute the above we would have to set it up as a limit,

$$\frac{d^{1/2}}{dx^{1/2}}\left[\frac{d^{1/2}}{dx^{1/2}}[1]\right] = \lim_{\alpha\to 0}\frac{d^{\alpha}}{dx^{\alpha}}\left[\frac{d^{1/2}}{dx^{1/2}}[1]\right] = \lim_{\alpha\to 0}\frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(-\frac{1}{2}-\alpha+1\right)}x^{-\frac{1}{2}-\alpha} = 0$$

□

Fractional Derivatives of Exponential Functions

Consider the natural exponential function, $f(x) = e^{cx}$, for a fixed real number c. The first few (integer ordered) derivatives of $f(x)$ are

$$f'(x) = ce^{cx}, \qquad f''(x) = c^2e^{cx}, \qquad f'''(x) = c^3e^{cx}$$

Thus,

$$f^{(n)}(x) = c^n e^{cx}$$

It then seems reasonable to say the following: for any positive real number α, the α^{th} order derivative of $f(x) = e^{cx}$ is

$$f^{(\alpha)}(x) = c^{\alpha}e^{cx}$$

This seems reasonable, but it will only be true if it satisfies the theorem above. We will first check the case for the one-half derivative

$$\frac{d^{1/2}}{dx^{1/2}}[e^{cx}] = c^{1/2}e^{cx}$$

Then,

$$\frac{d^{1/2}}{dx^{1/2}}\left[\frac{d^{1/2}}{dx^{1/2}}[e^{cx}]\right] = \frac{d^{1/2}}{dx^{1/2}}\left[c^{1/2}e^{cx}\right] = c^{1/2}\frac{d^{1/2}}{dx^{1/2}}[e^{cx}] = c^{1/2}c^{1/2}e^{cx} = ce^{cx}$$

That is, the half derivative of the half derivative is the whole derivative. More generally, for any real numbers α and β

$$\frac{d^\alpha}{dx^\alpha}\left[\frac{d^\beta}{dx^\beta}[e^{cx}]\right] = \frac{d^\alpha}{dx^\alpha}\left[c^\beta e^{cx}\right] = c^\beta \frac{d^\alpha}{dx^\alpha}[e^{cx}] = c^\beta c^\alpha e^{cx} = c^{\alpha+\beta} e^{cx} = \frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}[e^{cx}]$$

Therefore, we officially define the fractional derivative of the natural exponential function as

$$\boxed{\frac{d^\alpha}{dx^\alpha}[e^{cx}] = c^\alpha e^{cx}}$$

Note: For any exponential function $f(x) = b^x$, we can show that $f^{(n)}(x) = b^x(\ln b)^n$ for any positive integer n. Thus, we make a guess that for any positive real number α

$$\frac{d^\alpha}{dx^\alpha}[b^x] = b^x(\ln b)^\alpha$$

We will check that the above theorem holds for this definition,

$$\frac{d^\alpha}{dx^\alpha}\left[\frac{d^\beta}{dx^\beta}[b^x]\right] = \frac{d^\alpha}{dx^\alpha}\left[b^x(\ln b)^\beta\right] = (\ln b)^\beta \frac{d^\alpha}{dx^\alpha}[b^x] = (\ln b)^\beta b^x(\ln b)^\alpha = b^x(\ln b)^{\alpha+\beta} = \frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}[b^x]$$

Thus, we take the above proposed definition as the definition of the fractional derivative of an exponential function.

Note: We can write an arbitrary exponential function $f(x) = b^x$ in terms of the natural exponential function as follows

$$f(x) = b^x = e^{\ln(b^x)} = e^{x\ln b}$$

We can the compute fractional derivatives of the natural exponential function, so the derivative of order α of an arbitrary exponential function is

$$\frac{d^{\alpha}}{dx^{\alpha}}[b^x] = \frac{d^{\alpha}}{dx^{\alpha}}\left[e^{x\ln b}\right] = e^{x\ln b}(\ln b)^{\alpha} = b^x(\ln b)^{\alpha}$$

which agrees with the result for the arbitrary exponential function stated previously.

Example: Compute the three-sevenths derivative of $f(x) = 5^x$.

Solution:

$$\frac{d^{3/7}}{dx^{3/7}}[5^x] = 5^x(\ln 5)^{5/7}$$

Example: Compute the half- derivative of $f(x) = 5x^2 + e^{9x} + 4^x$.

Solution:

$$\frac{d^{1/2}}{dx^{1/2}}[5x^2 + e^{9x} + 4^x] = 5 \cdot \frac{8}{15\sqrt{\pi}}x^{3/2} + 9^{1/2}e^{9x} + 4^x(\ln 4)^{1/2}$$

$$= \frac{40}{15\sqrt{\pi}}x^{3/2} + 3e^{9x} + 4^x\sqrt{\ln 4}$$

□

Fractional Derivatives of the Sine and Cosine Function

Consider the first four derivatives of the sine function.

$$\frac{d}{dx}[\sin x] = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$\frac{d^2}{dx^2}[\sin x] = \frac{d}{dx}\left[\sin\left(x + \frac{\pi}{2}\right)\right] = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}\right)$$

$$\frac{d^3}{dx^3}[\sin x] = \frac{d}{dx}\left[\sin\left(\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}\right)\right] = \cos\left(\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}\right) = \sin\left(\left(\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}\right) + \frac{\pi}{2}\right)$$

Notice that each time we differentiate the sine function, the graph is shifted to the left by $\frac{\pi}{2}$ units. This same property holds for the cosine function. That is, for any natural number n,

$$\boxed{\begin{aligned} \frac{d^n}{dx^n}[\sin x] &= \sin\left(x + \frac{n\pi}{2}\right) \\[2em] \frac{d^n}{dx^n}[\cos x] &= \cos\left(x + \frac{n\pi}{2}\right) \end{aligned}}$$

These properties are used as motivation for the following definition. The example which follows the definition will demonstrate that the definition is reasonable.

Definition: For any $\alpha \geq 0$, the **derivative of order $\boldsymbol{\alpha}$** of the sine and cosine function are given by

$$\boxed{\begin{aligned} \frac{d^\alpha}{dx^\alpha}[\sin x] &= \sin\left(x + \frac{\alpha\pi}{2}\right) \\[2em] \frac{d^\alpha}{dx^\alpha}[\cos x] &= \cos\left(x + \frac{\alpha\pi}{2}\right) \end{aligned}}$$

Notes:

1. Recall the pattern that every fourth derivative of the sine (or cosine) function gets back to the original function. The above formulas demonstrate that based on the periodicity of the sine (or cosine) function,

$$\frac{d^4}{dx^4}[\sin x] = \sin\left(x + \frac{4\pi}{2}\right) = \sin(x + 2\pi) = \sin x$$

2. The above formulas for the fractional derivatives of the sine and cosine function show that differentiation of the function is equivalent to a horizontal shift of the graph. Thus differentiating a sine (or cosine) function with positive integer order corresponds to a shift, and now we are going in the other direction and saying given a shift, we can identify it with a derivative. Thus, for a shift other than a multiple of $\frac{\pi}{2}$, it is identified with a fractional derivative.

Example: Compute the following fractional derivatives

$$\frac{d^{2/3}}{dx^{2/3}}[\sin x] \qquad\qquad \frac{d^{1/3}}{dx^{1/3}}\left[\frac{d^{2/3}}{dx^{2/3}}[\sin x]\right]$$

Solution:

$$\frac{d^{2/3}}{dx^{2/3}}[\sin x] = \sin\left(x + \frac{\frac{2}{3}\cdot\pi}{2}\right) = \sin\left(x + \frac{\pi}{3}\right)$$

To compute a fractional derivative of this shifted sine function, we will have to use the identity for the sine of the sum of two angles,

$$\frac{d^{1/3}}{dx^{1/3}}\left[\frac{d^{2/3}}{dx^{2/3}}[\sin x]\right] = \frac{d^{1/3}}{dx^{1/3}}\left[\sin\left(x + \frac{\pi}{3}\right)\right] = \frac{d^{1/3}}{dx^{1/3}}\left[\sin(x)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\cos(x)\right]$$

$$= \frac{d^{1/3}}{dx^{1/3}}\left[\sin(x)\cos\left(\frac{\pi}{3}\right)\right] + \frac{d^{1/3}}{dx^{1/3}}\left[\sin\left(\frac{\pi}{3}\right)\cos(x)\right] = \frac{1}{2}\cdot\frac{d^{1/3}}{dx^{1/3}}[\sin(x)] + \frac{\sqrt{3}}{2}\cdot\frac{d^{1/3}}{dx^{1/3}}[\cos(x)]$$

$$= \frac{1}{2}\cdot\sin\left(x + \frac{\frac{1}{3}\cdot\pi}{2}\right) + \frac{\sqrt{3}}{2}\cdot\cos\left(x + \frac{\frac{1}{3}\cdot\pi}{2}\right) = \frac{1}{2}\cdot\sin\left(x + \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}\cdot\cos\left(x + \frac{\pi}{6}\right)$$

$$= \frac{1}{2}\left[\sin(x)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)\cos(x)\right] + \frac{\sqrt{3}}{2}\left[\cos(x)\cos\left(\frac{\pi}{6}\right) - \sin(x)\sin\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\sin(x) \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \cos(x)\right] + \frac{\sqrt{3}}{2}\left[\cos(x) \cdot \frac{\sqrt{3}}{2} - \sin(x) \cdot \frac{1}{2}\right]$$

$$= \frac{\sqrt{3}}{4}\sin(x) + \frac{1}{4}\cos(x) + \frac{3}{4}\cos(x) - \frac{\sqrt{3}}{4}\sin(x)$$

$$= \cos(x)$$

□

The previous example provides another verification that computing successive fractional derivatives whose orders add up to a positive integer, will give the integer ordered derivative. Also, the sum rule for fractional derivatives is nothing more than the addition of phase shifts.

Fractional Derivatives of Negative Order

How should we interpret a derivative of order -1? Since we want the differential operators to satisfy the property that the order of successive derivatives should add, we would want the following to be true

$$\frac{d^{-1}}{dx^{-1}}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^0}{dx^0}[f(x)] = f(x)$$

So for example, if we wanted to compute

$$\frac{d^{-1}}{dx^{-1}}[x]$$

what we really would be looking for is a function $f(x)$ which would satisfy the equation $\frac{d^{-1}}{dx^{-1}}\left[\frac{d}{dx}[f(x)]\right] = f(x)$. That is, the goal is to write

$$x = \frac{d}{dx}[f(x)]$$

Thus, we ask the question "what did we take the derivative of, to obtain x?". This can be a difficult question, and we will spend a great deal of time on it in the textbook.

Example: The derivative of order -1 of x is $\frac{1}{2}x^2$, since

$$\frac{d}{dx}\left[\frac{1}{2}x^2\right] = x$$

In fact, $\frac{1}{2}x^2 + 5,\ \frac{1}{2}x^2 - 3, \ldots$ are other possible choices for the derivative of order -1. As we will discuss in the textbook, there are infinitely many possible choices for the derivative of order -1, all of the form $\frac{1}{2}x^2 + c$ for a constant c. To verify this,

$$\frac{d}{dx}\left[\frac{d^{-1}}{dx^{-1}}[x]\right] = \frac{d}{dx}\left[\frac{1}{2}x^2 + c\right] = x$$

□

The following definition establishes an alternative notation for derivatives of negative integer order.

Definition: The derivative of order -1 is defined as

$$\boxed{\frac{d^{-1}}{dx^{-1}}[f(x)] = \int f(x)dx}$$

Similarly, for any natural number n,

$$\frac{d^{-n}}{dx^{-n}}[f(x)] = \underbrace{\int \ldots \int f(x)dx \ldots dx}_{n \text{ integrals}}$$

These differential operators satisfy the following property,

$$\frac{d^{-m}}{dx^{-m}}\left[\frac{d^{-n}}{dx^{-n}}[f(x)]\right] = \frac{d^{-m-n}}{dx^{-m-n}}[f(x)]$$

Note: Using this notation we can write the Fundamental Theorem of Calculus as

$$\frac{d^{-1}}{dx^{-1}}\left[\frac{d}{dx}[f(x)]\right] = f(x)$$

and

$$\frac{d}{dx}\left[\frac{d^{-1}}{dx^{-1}}[f(x)]\right] = f(x)$$

In this form it appears that differentiation has an inverse operation, that of differentiating with order -1.
Note: A negative order derivative of either a sine or a cosine function corresponds to a phase shift to the left.

Example: Compute the following derivative
Solution:

$$\frac{d^{-1.3}}{dx^{-1.3}}[7^x] = 7^x(\ln 7)^{-1.3}$$

□

The fractional derivative formulas for the non-negative order derivatives of the sine, cosine and exponential functions are defined for all real numbers and so the negative order derivatives are computed using these same formulas.

Exercises

---

1. Using the definition of the fractional derivative for a power function, show the following

$$\frac{d^0}{dx^0}[x^k] = x^k$$

2. Using the properties of the gamma function, find the value of $\Gamma\left(\frac{3}{2}\right)$, $\Gamma\left(\frac{5}{2}\right)$ and $\Gamma\left(\frac{7}{2}\right)$.

3. Let $f(x) = x^2$. Use Wolfram Alpha to find decimal approximations for the values of the gamma function for the following fractional derivatives. Approximate the decimals to 4 decimal places

$a)\ \frac{d^{1/4}}{dx^{1/4}}[x^2]$ $b)\ \frac{d^{1/3}}{dx^{1/3}}[x^2]$ $c)\ \frac{d^{1/2}}{dx^{1/2}}[x^2]$

$d)\ \frac{d^{2/3}}{dx^{2/3}}[x^2]$ $e)\ \frac{d^{3/4}}{dx^{3/4}}[x^2]$ $f)\ \frac{d^{4/5}}{dx^{4/5}}[x^2]$

4. Using either your calculator, or http://www.wolframalpha.com/, graph the function $f(x) = x^2$ along with its first derivative and all of the fractional order derivatives you obtained in exercise #3.

5. Compute the fractional derivatives, then graph them all on the same plot along with $f(x) = \sin x$ and its first derivative

$a)\ \frac{d^{1/4}}{dx^{1/4}}[\sin x]$ $c)\ \frac{d^{1/2}}{dx^{1/2}}[\sin x]$ $e)\ \frac{d^{3/4}}{dx^{3/4}}[\sin x]$

$b)\ \frac{d^{1/3}}{dx^{1/3}}[\sin x]$ $d)\ \frac{d^{2/3}}{dx^{2/3}}[\sin x]$ $f)\ \frac{d^{4/5}}{dx^{4/5}}[\sin x]$

6. Using decimal approximations (computed with wolfram alpha), find the two-fifths derivative of $f(x) = 3x^2$, then find the three-fifths derivative of the result to verify that you would get the first derivative.

7. Using decimal approximations (computed with wolfram alpha), find the one-fifth derivative of $f(x) = 2x^4$, then find the two-thirds derivative of the result, then find the two-fifteenths derivative of that result to verify that you would get the first derivative. That is, compute the following

$$\frac{d^{2/15}}{dx^{2/15}}\left[\frac{d^{2/3}}{dx^{2/3}}\left[\frac{d^{1/5}}{dx^{1/5}}[2x^4]\right]\right]$$

8. Define a function φ as follows

$$\varphi(\alpha) = \frac{d^\alpha}{dx^\alpha}[x^2]\bigg|_{x=1}$$

That is, the independent variable α for the function φ is the order of the derivative. For example

$$\varphi(0) = \frac{d^0}{dx^0}[x^2]\bigg|_{x=1} = x^2|_1 = 1$$

and

$$\varphi(1) = \frac{d^1}{dx^1}[x^2]\bigg|_{x=1} = 2x|_1 = 2$$

Using Wolfram Alpha to do the decimal approximations, compute $\varphi(0.1), \varphi(0.2), \varphi(0.3), \ldots \varphi(0.9)$. Then plot these points to make a graph for $\varphi(\alpha)$ on the interval $0 \leq \alpha \leq 1$.

9. Define

$$\varphi(\alpha) = \frac{d^\alpha}{dx^\alpha}[x^2]\bigg|_{x=2}$$

Compute $\varphi(0)$ and $\varphi(1)$ by hand and then use Wolfram Alpha to do the decimal approximations, compute $\varphi(0.1), \varphi(0.2), \varphi(0.3), \ldots \varphi(0.9)$. Then plot these points to make a graph for $\varphi(\alpha)$ on the interval $0 \leq \alpha \leq 1$.

10. Define

$$\varphi(\alpha) = \frac{d^\alpha}{dx^\alpha}[x^2]\bigg|_{x=c}$$

a) Show that we can write

$$\varphi(\alpha) = \frac{2c^2}{c^\alpha \Gamma(3-\alpha)}$$

b) Writing the function $\varphi(\alpha)$ as in part a will allow you to type the function into Wolfram Alpha, and to graph it. To type it in, first choose a value for c (say, $c = 5$) and then to graph type it in as

graph 2(5^2)/((2^x)*gamma(3-x)) from x=0 to x=1

where we are using x as the independent variable instead of α.

11. Define

$$\varphi(\alpha) = \frac{d^\alpha}{dx^\alpha}[x^3]\bigg|_{x=c}$$

a) Show that we can write

$$\varphi(\alpha) = \frac{6c^3}{c^\alpha \Gamma(4-\alpha)}$$

b) Let $c = 1$ and then graph $\varphi(\alpha)$ for $0 \le \alpha \le 1$.

c) Let $c = 1$ and then graph $\varphi(\alpha)$ for $0 \le \alpha \le 2$.

d) Let $c = 1$ and then graph $\varphi(\alpha)$ for $0 \le \alpha \le 3$.

e) Let $c = 1$ and then graph $\varphi(\alpha)$ for $0 \le \alpha \le 4$.

12. Define

$$\psi(\alpha, x) = \frac{d^\alpha}{dx^\alpha}[x^2]$$

a) Show that $\psi(0, x) = x^2$

b) Show that $\psi(1, x) = 2x$

c) Find $\psi(0.1, x), \psi(0.2, x), \dots, \psi(0.9, x)$. Then, graph $\psi(0, x), \psi(0.1, x), \psi(0.2, x), \dots, \psi(0.9, x), \psi(1, x)$ on the same plot for $0 \le x \le 5$.

$d)$ Suppose we were to graph $\psi(\alpha, x)$ for $0 \leq \alpha \leq 1$ and for $0 \leq x \leq 5$. What would this look like?

13. Verify the following

$a)\dfrac{d^{-1}}{dx^{-1}}[\sin(x)] = -\cos(x)$

$b)\dfrac{d^{-1}}{dx^{-1}}\left[\dfrac{2}{5}e^{3x}\right] = \dfrac{2}{15}e^{3x}$

$c)\dfrac{d^{-2}}{dx^{-2}}[x^2] = \dfrac{1}{12}x^4$

14. The function $g(x) = \sin\left(x + \frac{\pi}{5}\right)$ is the derivative of what order of the sine function? Similarly, identify the order of the derivative for the following: $g(x) = \sin\left(x + \frac{3\pi}{4}\right)$, $g(x) = \sin\left(x + \frac{1}{3}\right)$, $g(x) = \cos\left(x + \frac{2\pi}{7}\right)$ and $g(x) = \cos\left(x + \frac{\sqrt{2}}{11}\right)$.

The Random Secretary: A Probability Challenge for All Levels and an Exercise in Critical Thinking in Mathematics

Paul Bedard

Teaching statistics is an invigorating challenge. Unlike the student population in tracked courses like the calculus sequence, there is a lot of diversity in mathematical background. Some students have just barely finished intermediate algebra and are still shaky about fractions and solving linear equations. Others have finished differential equations. Keeping the class interesting and challenging for the latter group while still always approachable for the former can be difficult. I am reminded of the early American one room schoolhouse, in which several grades were taught at the same time. The teacher might write an assignment on the chalkboard for the fourth graders to attend to, while speaking to the first graders. Since the divisions in my student population are more invisible, and since I have the same overall objectives for all students, my solution to this dilemma must be more elegant.

One way to keep the upper tier challenged is to present multiple versions of the same concepts. While I have the same learning goals for all students, the way the students approach the problems may differ. Here is an example. I recently explained the exponential probability distribution. This is a continuous distribution (values of the random variable come from an uncountably infinite set.) The exponential distribution models waiting times. If you are in line at a drive-through restaurant, what is the likelihood that you will wait for five minutes? It turns out that the required parameter is mean waiting time. What is the average amount of time customers wait? Let's suppose this average is three minutes.

For the calculus-level students, I mention that the answer to this question is a definite integral. The probability of waiting for five minutes or longer is

$$\int_5^{\infty} \frac{1}{3} e^{-x/3}\, dx$$

(The interested reader is encouraged to evaluate this integral. It will also be instructive to verify that this does indeed represent a probability distribution by verifying that

$$\int_0^{\infty} \frac{1}{3} e^{-x/3}\, dx = 1)$$

For a student in calculus 2 or beyond, the problem above is not very difficult. It is an excellent review, and it displays for these students the usefulness of what they learned in their other courses, and the connectivity of mathematics. Indeed, integral calculus is the cornerstone of modern probability theory, but most introductory courses don't even mention it; the textbooks certainly don't.

The integral approach leaves most of the class gasping for air, of course, so I don't emphasize it. After presenting it, I immediately proceed to discuss the exponential distribution from an analytical thinking perspective. Try these problems and see what you think. (All answers appear at the end of this article.)

If the mean waiting time in a line is 3 minutes, the probability of waiting 5 or more minutes is about 18.9%. The probability of waiting between 4 and 5 minutes is about 7.5%.

a) What is the probability of waiting less than 5 minutes?

b) What is the probability of waiting less than 4 minutes?

c) Is the exponential distribution symmetrical, in the sense that intervals less than the mean (like from 1 to 2 minutes in this case) have the same probability as equal intervals the same amount greater than the mean (like 4 to 5 minutes)? This problem requires some intuition. Calculus students: can you prove your intuition?
Non calculus students: what evidence do you have that your intuition is correct? (This is a critical thinking question.)

d) Estimate the probability of waiting 6 or more minutes.

e) Estimate the probability of waiting 10 or more minutes.

f) Explain the word "distribution" in this context. What is being distributed?

Note that some of the above questions require different kinds of thinking than others. Students are asked to intuit and to estimate, to look for evidence when they can't prove, and to put vocabulary in context. These are goals I set for all my mathematics students. In this example, students at different levels can approach the problem in different ways, but it is approachable by all. I consider these to be moderately

“deep” problems. Rather than coasting over the surface of the subject with memorized formulas or by following rigid algorithms, the students have to think for themselves and try to grasp what is going on behind the curtain, so to speak. I find this approach surprisingly effective. I believe that many students find mathematics ‘boring’ (a bizarre conclusion, to be sure) because following recipes step by step without ever looking at or sniffing what you are putting into the bowl is the work of a drudge.

I love a challenge. One of my favorite teaching techniques is to challenge the students to seemingly impossible tasks. The harder it is, the more startling they find it that I would ask them to try it. The more I tease them with the idea that it is too much for them, the more they want to prove me wrong. (Actually, they want to prove me right, since I always make it clear that I do in fact expect them to succeed. The psychology here is interesting, I think.)

I was aggravated by a statement in the statistics text I use that the explanation for a certain conclusion was “beyond the scope of this course – way beyond.” So, I taught the problem! I must admit, when the problem appeared on the test, several witty students answered by saying, “This problem is beyond the scope of my abilities…way beyond!” I was compelled to assign a small amount of partial credit; after all, this came directly from the text.

The problem was set up in the textbook as an interesting marginal note, and not solved. If a secretary has 50 different letters and 50 addressed envelopes, and he places letters in envelopes at random, what is the probability that at least one letter ends up in the right envelope?

Before looking at the answer, please think about this. What sort of answer would you expect? Is it very unlikely that any letter ends up in the right place? Or is it nearly certain that “at least one” placement is correct? Why do you think so?

Triola provides the answer by saying that the probability is 0.632 (63.2%). He then goes on to make a startling assertion: “In fact, the probability is 0.632 even if there are a million letters and a million envelopes.”

This conclusion is counterintuitive. That means that it challenges our “common sense” way of looking at things. Surely, with so many more possibilities, the probability must change significantly!

I decided to use this problem to stimulate some critical thinking. First, what does the assertion imply that is not stated? I think one implication is clear: that the probability is the same, regardless of how many letters and envelopes. It is unlikely that 0.632 happens to be the correct value when there are 50 letters, and also when there are one million, but not for the values in between. It seems likely (does it not?) that Triola chose the value “one million” for effect, not because that particular value holds any great significance. Also,

the word "even," ("even if there are a million envelopes") suggests this.

Another fair assumption is that the value 0.632 is exact, or, if rounded, that the probabilities for 50 letters and one million letters are exactly the same. In fact, Triola doesn't say "about 0.632." I think this at least allows the interpretation that the value is exact. The statement itself is not very exact, from a mathematical viewpoint.

Now, at this point in the course, I have discussed the difference between discrete values (how many eggs are in your refrigerator?) and continuous values (how tall are you, to the utmost level of precision in measurement?) many times. I think the students ought to be surprised at the implication that the probability is exactly 0.632; that is, that it is equal to 0.632000…

Of course, this probability is not exactly 0.632. One fair assumption from the wording of the assertion fails. What about the other? Is the probability exactly the same for all numbers of envelopes? What do you think?

I hope you said, "Of course it isn't!" This is easy to show. Let's use a Polya principle: replacing a problem with a simpler problem. Suppose you have one letter and one envelope, and you place the letter in the envelope "at random." What is the probability that you get at least one letter in the correct envelope? Of course, you are certain to do so. The probability is therefore 1, or 100%. This value, as should be clear, is significantly different from 0.632.

So far, I think the students have learned a lot. Let's go on and try the next value, the probability when there are 2 letters and 2 envelopes. To organize our work, lets call the letters *a* and *b* and the envelopes *A* and *B*. The sample space (the set of all "atomic" or irreducible states) is therefore represented by pairings of every possible lower case letter paired with every possible upper case letter. The sample space in this case is {*aA bB, aB bA*}.

Each of these "events" is equally likely. Only one of them – the first- satisfies the condition "at least one letter is in the correct envelope." So, the probability that at least one letter ends up in the correct envelope is 0.5. (Note that I didn't say "or 50%" this time. Are you getting the idea?)

Now let's not lose our momentum. Set up the sample space for three letters and three envelopes.

Here it is: {*aA bB cC, aA bC cB, aB bA cC, aB bC cA, aC bB cA, aC bA cB*}

Did you get all those? How did you organize your work to avoid missing any possibilities? Describe how I organized mine.

There are six atomic states. How many satisfy our condition? Which are they? The probability of at least one letter in a correct envelope is 4/6, or 0.666…

Did the number 0.666… excite you? It is closer to the 0.632 we were told about.

Let's list our answers in a sequence. A sequence is a list of numbers, separated by commas. Here is our "envelope sequence": 1, 0.5, 0.666.

How many terms would be in the sequence if we solved for every possible number of envelopes? Of course, this would be an infinite sequence.

Please compare the following pair of infinite sequences:

1,2,3,4,5,… (1)

0.9, 0.99, 0.999, 0.9999, … (2)

What do you think is the most important difference between sequence (1) and sequence (2)? Let me help you a little. Consider the differences between consecutive entries. Are they getting smaller?

Sequence (1) is said to "diverge." To oversimplify a little, this means it isn't approaching any fixed value. Sequence (2) "converges." What value are the entries approaching? Will this value, called the "limit" of the sequence, ever appear as an entry in the sequence?

Now, with this little bit of background in sequences, let's look again at our sequence of probabilities.

1, 0.5, 0.666, ?, ?, ?, … (3)

Does it appear that this sequence will converge, or diverge? What evidence are you using to make your claim?

Suppose you decided that the sequence (3) is convergent. What is its limit? Of course, the answer we are expecting is 0.632. Here is an interesting thought: how does the manner in which (3) approaches 0.632 differ from the manner in which (2) approaches 1?

Perhaps all these questions are a little premature.

Another problem for you:

g) Please set up the sample space for four letters and four envelopes, and find the fourth entry in (3). (The answers appear at the end of the article.)

I would like to pursue this problem a little further, before I draw my final conclusions and say a few more things about students and challenges. This envelope problem is essentially identical to a game invented by the British mathematician Arthur Cayley called "Mousetrap." Details of this game can be found at Wolfram Mathworld, http://mathworld.wolfram.com/Mousetrap.html

I will need to introduce a new term.

“Factorial”, represented by the symbol “!”, is very common in probability. I will give an ostensive definition of factorial (providing examples rather than stating what something means in other words.)

$1! = 1$

$2! = 2 \times 1$

$3! = 3 \times 2 \times 1$

…

$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

Please consider a few problems.

h) What is $100!/98!$

i) What is $n!/(n-1)!$

j) Which is larger for large values of n: $n!$ or n^2? n^{100} ? 2^n ? 100^n?

Factorials are an important tool for counting things. Here’s an example: how many ways are there to arrange the letters in the word “mousetrap?” To count this very large number of arrangements, consider the fact that we must first select a letter to go in the first position. There are 9 letters in “mousetrap,” so there are 9 ways to select the first letter. Once we have selected a letter to occupy the first position, we have only 8 letters remaining as choices for the second position. In the same way, there will be 7 letters for us to select from to fill the third position, and so on until we reach the ninth and final slot, which can be filled in only one way since by then we will have used 8 of the 9 letters to fill the first 8 positions. Therefore, the number of arrangements is $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, or 9!. Use a calculator to see that this is indeed a large number.

Consider how factorials relate to the random letter problem. Take another look at how I set up the sample spaces. Notice that I always list the letters in alphabetical order. The envelopes are therefore the only component of each element that changes. So, for instance, the element *aB bA cC* can be restated as *BAC*. Therefore, the total number of ways to fill 3 envelopes is identical to the total number of arrangements of the letters *ABC*. This is 3!. The total number of ways to fill *n* envelopes is *n!*.

n! is the size, or cardinality, of the sample space for n letters. There are 5! Or 120 ways to fill 5 envelopes with 5 letters. But, how many of those include at least one letter in the correct envelope?

Suppose we have three envelopes and three letters. What is the probability that at least one of the letters is correctly placed? (We have done this constructively by listing the sample space, but this becomes too difficult. We want to see if there is a pattern we can apply to find the values for any value of *n*.) Let’s

start with the probability that letter *a* ends up correctly placed in envelope *A*. There is only one way to accomplish this, but there are two letters remaining, and therefore 2! ways that those can be placed. 1 times 2! is 2!, so there are 2! ways that letter *a* can be correctly placed in envelope *A.* But, the same is true for both *b* and *c*. So there are 2! + 2! + 2! ways that in total that either letter *a* or letter *b* or letter *c* can be placed correctly.

But this is a total of 6 ways! There are, recall, 3! or 6 total possibilities. That would make the likelihood of either letter *a* or letter *b* or letter *c* being placed correctly 6/6, or 1. Certainty! That can't be right.

Of course, it isn't. We have overlooked something important. We have double or even triple counted some possibilities. For instance, one of the 2! "letter *a* was correctly placed" possibilities is *aAbBcC*, or *ABC* to use our shorthand. This possibility was counted three times, in fact. We want to subtract it away. But, let's do this more systematically. Let's consider all the *aA* possibilities that also include *bB*. We can think of this as the intersection of *"a* is placed correctly" with "*b* is placed correctly." (In this case, there is only one such possibility and it is the triple overlap we have already mentioned, but remember, we want to apply the method we are inventing here to larger values of *n* where that will not be true. So, the overlap of *"a* is placed correctly" with "*b* is placed correctly" includes 1 possibility, so we should subtract $1/3!$ from our total.

But we must do this also for the overlap of *"a* is placed correctly" with *"c* is placed correctly", and again for the overlap (or intersection) of *"b* is placed correctly" with *"c* is placed correctly". So, we have $1 - 3/3!$, which simplifies to $1 - 1/2!$ (why?)

But this is still not correct, because we have subtracted away three overlaps, so the triple overlap, *aAbBcC*, or *ABC*, which was counted three times, has now been subtracted away three times! We need to add it back. The probability of *aAbBcC* is $1/3!$.

So now, we are ready to unveil our masterpiece. When randomly filling three envelopes with three letters, the probability that at least one letter is correctly placed is

$$1 - \frac{1}{2!} + \frac{1}{3!}$$

This tallies with our brute force conclusion. Does this help us with larger values of *n*? It actually does. For four letters, we get

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}, \text{ or } 0.625$$

For 5 letters,

$$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!},\ \text{ or }\ 0.6333\ldots$$

and for *n* letters,

$$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}-\cdots+(-1)^{n+1}\frac{1}{n!}$$

(The "$(-1)^{n+1}$" guarantees that the terms in which *n* is odd will be positive, and the terms where *n* is even will be negative.)

Please note that this also works on our original examples. If we go back to the two letter case, we get

$$1-\frac{1}{2!},\text{or } 0.5$$

And in the case of one letter, we get

$$1$$

Now, I have not proven that this pattern holds. To do so, I could utilize a technique of proof called mathematical induction. I will refrain from ironically saying that this would be well beyond the scope of my article, but I will state that it isn't very important to my overall point here, so I will ask you to look it up on your own or take my word for it, just this once.

So, now let's go back to the idea of the limit of a sequence. The "envelope sequence" (3) can now be written

$$1, 1-\frac{1}{2!}, 1-\frac{1}{2!}+\frac{1}{3!}, \ldots$$

To what value does this sequence converge? It converges to

$$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}-\cdots$$

This is called an "infinite series." It "goes on forever" only in the sense that, **written in this particular manner**, there is no final term. The sum of all these infinitely many terms, however, is quite

finite. (Does this seem counterintuitive? Pause and reflect!)

The sum, as calculus students could tell us, is $1 - 1/e$, where *e* represents a special, irrational number (like pi, its digits also "go on forever" without forming any repeating patterns.)

Since *e* is irrational, so is $1 - 1/e$, and we have proven that one implication of the Triola assertion is false. The value 0.632 is a rounded value! No two values of *n* yield exactly the same probability value.

This wonderful, rich and dense example has led to a discussion of infinite sequences, infinite series, convergence, discrete and continuous data, counterintuitive conclusions, rational and irrational values, and critical thinking. As I said earlier, I love a challenge, and making elementary mathematics more challenging for students yields rich rewards.

Answers to exercises.

a) 100% – 18.9%, or 81.1% (0.811)

b) 1 – (0.189 + 0.075), or 0.736

c) No, the distribution is far from symmetrical. Intervals of the same width are less likely the further along they are.

d) About 13.5%.

e) About 3.6%

f) There is a 100% probability of waiting for some amount of time, including zero. This 100% is distributed among subintervals.

g) We will keep all letters in alphabetical order, so the only thing we need to list is the envelopes. In other words, *aBbCcAdD* will be written as *BCAD*.

ABCD, ABDC, ACBD, ACDB, ADBC, ADCB

BACD, BADC, BCAD, BCDA, BDAC, BDCA

CABD, CADB, CBAD, CBDA, CDAB, CDBA

DABC, DACB, DBAC, DBCA, DCAB, DCBA

(My student Lena Szuminski had a beautiful color-coded chart for keeping track of these permutations.)

h) 9900

i) *n*

j) For *n* sufficiently large, *n*! is larger than any of these.

Communicating in Statistics on Day 1

Marie St. James

The first day of my Introduction to Statistics course involves introductions, distribution of the syllabus and expectations. After all of the paperwork is completed, there is plenty of time for an activity in class. Activities and real data are vital in the context of an elementary statistics course if we hope to engage our students in critical thinking. As instructors, we are aware of the old saying:

Tell me something, and I will forget.
Show me and I will remember.
Involve me, and I will learn.

The activity involves a survey of the 50 states in the United States. Students are given a handout with the 50 states listed. The instructions are "Put a check mark next to each state you have visited", and then one question "How many states have you visited". These are relatively simple instructions that can generate engaging conversations. In addition, we can accomplish a few content objectives on the first day of class.

My primary objective for this activity is *communication*. I give the students time to complete the survey. Then instead of tallying the surveys myself and reporting back the data, I engage the students right away. We tally the data together verbally. I call out the state and have one or two students act as the "counters" to report how many students visited that state. The data is collected on the SMART Board so that everyone knows the score. The activity uses student data that is not too intrusive and lets me establish on the first day that everyone will join the conversation. After everyone reports their own data, we do the math, albeit, a minimal amount of calculations.

We observe and ask some questions:

1. What is the average number of states visited?
2. What is the average number of visitors per state?
3. We discuss the difference between the first two questions.
4. What state had the most visitors? The least? Why?
5. Why are some states more popular? In other words, why are there more visitors to Ohio than New Mexico?

6. Is this a good survey?
7. Is the data accurate? Could our subjects lie? *Would* our subjects lie?
8. Is the data significant?
9. Who could use the data? Travel agents?
10. Is the data important in the scheme of larger world issues?

An auxiliary objective to the activity is to get students to understand how to design an experiment. Here we might include context, source of data and sampling methods. Typically though, before I get to question #7 above, a student will ask "how are we defining what it means to visit the state"? Students will want to argue a definition that might include an overnight stay or just driving through the state. There are always interesting definitions. The conversation can be quite animated.

The definition decided upon is not the goal here, but rather, the students need to see that the initial instructions did not include a definition. Thus, is this a good survey? Now we have some new vocabulary and topics to discuss:

- What makes a good survey/experiment?
- What do we need to define?
- What are the parameters?
- What type of data are we collecting within our survey? We can include nominal, ordinal, interval, ratio, qualitative, quantitative, etc.
- How do we define "visited the state"?
- Who completed the survey? The population of the class did in this case. But what about if we needed a larger population, or would we choose a sample of people?
- What type of samples could we choose?

Through this simple activity, I can introduce many of the topics covered in the first few sections of the statistics textbook. Enough, I believe, so that I can assign to the students to read those sections with better understanding. We have utilized non-intrusive personal data to generate an interesting conversation. It sets the precedent that I will engage the students in the coursework throughout the semester.

Finally, this activity also lets the students get to know a little bit about me. I share my family's definition of what qualifies as a state visit. This was defined many years ago on a car trip to Florida, by my two daughters who were 7 and 4 years old at the time. Just so you know now also, it involves a rest stop.

Advanced Integration Techniques

Nick Goins

The following consists of three integration techniques which could be introduced in the introductory calculus courses. The first topic discusses how to write a function into its even-odd decomposition, and then uses this along with symmetry to evaluate integrals which otherwise seem impossible to compute. This technique would fit in most naturally to the second semester of the calculus sequence along with the traditional integration techniques. The second topic, Feynman integration, requires a basic understanding of partial derivatives and while this would most naturally fit into the third semester of the sequence, it could be introduced in the second semester if the basic computational methods of partial derivatives are briefly discussed. The last topic, Weierstrass substitution, discusses how to convert a trigonometric integrand into a rational integrand and this would go well with trigonometric substitution, which converts an algebraic integrand into a trigonometric one.

1. THE EVEN-ODD DECOMPOSITION OF A FUNCTION AND INTEGRALS OVER A SYMMETRIC INTERVAL ABOUT THE ORIGIN

In what follows we will describe how to find what is called the even-odd decomposition of a function and tie that in with integrals over a symmetric interval about the origin. Note that some of the integrals in this section can only be computed as a definite integral. That is, we can evaluate the integral but we cannot find an antiderivative for the function (we will, however, find an antiderivative for the even part of the function).

The Even – Odd Decomposition of a Function

For any function $f(x)$, we can write

$$f(x) = \frac{f(x) - f(-x)}{2} + \frac{f(x) + f(-x)}{2}$$

As we will verify, the expression on the left is odd function and the expression on the right is an even function. Based on this, we make the following definition.

Definition: For any function $f(x)$, the **even part** of $f(x)$ is the following function

$$\boxed{f_e(x) = \frac{f(x) + f(-x)}{2}}$$

and the **odd part** of $f(x)$ is the following function

$$\boxed{f_o(x) = \frac{f(x) - f(-x)}{2}}$$

Example: Show that $f_e(x)$ is an even function.

Solution:

$$f_e(-x) = \frac{f(-x) + f\big(-(-x)\big)}{2} = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x)$$

□

Theorem: Any function can be written as the sum of an even function and an odd function.

Proof: $f(x) = f_e(x) + f_o(x)$.

Example: Find the even part of $f(x) = x^3 + x^2 + x - 2$. Then find the odd part.

Solution: The even part is

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{(x^3 + x^2 + x - 2) + ((-x)^3 + (-x)^2 + (-x) - 2)}{2}$$

$$\frac{x^3 + x^2 + x - 2 - x^3 + x^2 - x - 2}{2} = \frac{2x^2 - 4}{2} = x^2 - 2$$

The odd part is,

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \frac{(x^3 + x^2 + x - 2) - ((-x)^3 + (-x)^2 + (-x) - 2)}{2}$$

$$\frac{x^3 + x^2 + x - 2 + x^3 - x^2 + x + 2}{2} = \frac{2x^3 + 2x}{2} = x^3 + x$$

□

Notes:

1. This example provides evidence for the fact that the even portion of a polynomial function consists of the even degree terms and the odd part of the polynomial function consists of the odd degree terms. You can use this in the exercises when asked to find the even and odd part of a polynomial.

2. The even part of the natural exponential function is the hyperbolic cosine function and the odd part of the natural exponential function is the hyperbolic sine function.

Example: Find the even part of $f(x) = \sqrt[3]{2x+1}$. Then find the odd part.

Solution: The even part is

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{\sqrt[3]{2x+1} + \sqrt[3]{-2x+1}}{2}$$

The odd part is,

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \frac{\sqrt[3]{2x+1} - \sqrt[3]{-2x+1}}{2}$$

□

Example: If $f(x)$ is an even function, show that $f(x) = f_e(x)$ and that $f_o(x) = 0$.

Solution: We use the basic property of an even function (i.e., $f(-x) = f(x)$)

$$f(x) = f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(x)}{2} = f_e(x) \ + \frac{0}{2}$$
$$= f_e(x)$$

Notice that within the above calculation, we obtained $f_o(x) = 0$.

□

Now, we will begin to tie the above discussion in with derivatives and integrals. First, we will show the relationship between the two different types of symmetries (i.e., symmetry with respect to the y-axis and symmetry with respect to the origin) and a function with its derivative.

Theorem: The derivative of an even function is an odd function and the derivative of an odd function is an even function.

Proof: We will prove the first statement. If $f(x)$ is an even function, then we know that $f(-x) = f(x)$ for every x in its domain. Then, differentiating both sides of this equation we get

$$\frac{d}{dx}[f(-x)] = \frac{d}{dx}[f(x)] \qquad \Rightarrow \qquad -f'(-x) = f'(x) \qquad \Rightarrow \qquad f'(-x) = -f'(x)$$

That is, $f'(x)$ is an odd function.

□

The proof that the derivative of an odd function is even is similar to the above.

Example: The derivative of $f(x) = x^2$ is $f'(x) = 2x$ and the derivative of $f(x) = \sin x$ is $f'(x) = \cos x$, which provides simple examples of the even and odd result corresponding to a function and its derivative in the previous theorem.

Note: Suppose $F(x)$ is an antiderivative of $f(x)$ and suppose that $f(x)$ is an odd function. Then we have the following

$$F'(x) = f(x) \qquad \text{and} \qquad f(-x) = -f(x)$$

From this, what symmetry would $F(x)$ have? Since f is odd and $f = F'$, we see that F' is also odd. If F was an odd function, then F' would be even by the above theorem. Therefore, F must be an even function, since its derivative is odd.

This says that an antiderivative of an odd function is an even function.

Theorem: If $f(x)$ is an odd function, then the following is true for any value of a,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Proof: This is an exercise.

Corollary: For any function $f(x)$, for an integral over an interval of integration which is symmetric with respect to the origin, the integrand can be replaced with the even part of the function. That is,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f_e(x)\,dx$$

Example: Compute the following integral

$$\int_{-1}^{1} -2x^7 + 3x^5 + 3x^2 + 9x - 2\,dx$$

Solution: Since the interval of integration is symmetric with respect to the origin, we can replace the integrand with its even part. Thus,

$$\int_{-1}^{1} -2x^7 + 3x^5 + 3x^2 + 9x - 2\,dx = \int_{-1}^{1} 4x^2 - 2\,dx = \left[\frac{4}{3}x^3 - 2x\right]\Bigg|_{x=-1}^{1}$$

$$= \left(\frac{4}{3}(-1)^3 - 2(-1)\right) - \left(\frac{4}{3}(1)^3 - 2(1)\right) = \frac{4}{3}$$

□

Example: Compute the following integral

$$\int_{-2}^{2} \frac{x^4}{5^x + 1}\, dx$$

Solution: By the above corollary, we can replace the integrand with its even part, since the interval $[-2, 2]$ is symmetric with respect to the origin. The even part of the integrand is

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{1}{2}\left[\frac{x^4}{5^x + 1} + \frac{(-x)^4}{5^{-x} + 1}\right] = \frac{1}{2}\left[\frac{x^4}{5^x + 1} + \frac{x^4}{5^{-x} + 1}\right] = \frac{x^4}{2}\left[\frac{1}{5^x + 1} + \frac{1}{5^{-x} + 1}\right]$$

Then, we can combine the terms inside the brackets into a single term,

$$= \frac{x^4}{2}\left[\frac{5^{-x} + 1}{(5^x + 1)(5^{-x} + 1)} + \frac{5^x + 1}{(5^x + 1)(5 -^{x} + 1)}\right] = \frac{x^4}{2}\left[\frac{5^{-x} + 1 + 5^{-x} + 1}{(5^x + 1)(5^{-x} + 1)}\right] = \frac{x^4}{2}\left[\frac{5^{-x} + 5^{-x} + 2}{(5^x + 1)(5^{-x} + 1)}\right]$$

Distributing the denominator gives,

$$= \frac{x^4}{2}\left[\frac{5^{-x} + 5^{-x} + 2}{1 + 5^x + 5^{-x} + 1}\right] = \frac{x^4}{2}[1] = \frac{x^4}{2}$$

Thus, the above algebra gives the following,

$$\int_{-2}^{2} \frac{x^4}{5^x + 1}\, dx = \int_{-2}^{2} \frac{x^4}{2}\, dx = \frac{32}{5}$$

□

2. FEYNMAN INTEGRATION

Before we start looking at the integrals in this section, we first need to introduce a technique and some notation regarding derivatives of multivariable functions.

Definition: For a function with more than one variable, $f(x, y)$ for example, the **partial derivative of $f(x)$ with respect to x** is the derivative of the function f in which we interpret y (or more precisely, every variable other than x) as a constant. This derivative is denoted

$$\frac{\partial f}{\partial x}$$

Similarly, we can define the partial derivative of $f(x)$ with respect to y, and write this as

$$\frac{\partial f}{\partial y}$$

Example: Let $f(x, y) = x^2y + 5xy + y$. Compute the partial derivative of f with respect to x, then with respect to y.

Solution: To compute the partial derivative of $f(x)$ with respect to x, we will differentiate the function and treat y as a constant. That is,

$$\frac{\partial f}{\partial x} = 2xy + 5y$$

Similarly,

$$\frac{\partial f}{\partial y} = x^2 + 5y + 1$$

□

Theorem: Differentiation Under the Integral: For the function defined by

$$I(\alpha) = \int_a^b f(x, \alpha)\, dx$$

the derivative of I is given by

$$I'(\alpha) = \frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$$

Example:

$$\frac{d}{d\alpha}\int_0^\infty e^{-\alpha^2x^2}\,dx = \int_0^\infty \frac{\partial}{\partial\alpha}\left[e^{-\alpha^2x^2}\right]dx = \int_0^\infty -2\alpha x^2 e^{-\alpha^2x^2}\,dx = -2\alpha\int_0^\infty x^2 e^{-\alpha^2x^2}\,dx$$

Example: Consider the following definite integral,

$$\int_0^\infty \frac{x\sin(mx)}{x^2+\alpha^2}\,dx = \frac{\pi}{2}e^{-m\alpha}$$

Treat α as a parameter and differentiate with respect to α. Then determine the value of the following integral

$$\int_0^\infty \frac{x\sin(3x)}{(x^2+4)^2}\,dx$$

Solution:

$$\frac{d}{d\alpha}\left[\int_0^\infty \frac{x\sin(mx)}{x^2+\alpha^2}\,dx\right] = \frac{d}{d\alpha}\left[\frac{\pi}{2}e^{-m\alpha}\right] \qquad \Rightarrow \qquad \int_0^\infty \frac{\partial}{\partial\alpha}\left[\frac{x\sin(mx)}{x^2+\alpha^2}\right]dx = \frac{d}{d\alpha}\left[\frac{\pi}{2}e^{-m\alpha}\right]$$

Computing the derivatives gives

$$\int_0^\infty \frac{-2\alpha x\sin(3x)}{(x^2+\alpha^2)^2}\,dx = \frac{-m\pi}{2}e^{-m\alpha}$$

Since -2α is a constant with respect to the integral (whose variable of integration is x), we can pull that constant out and divide by it,

$$\int_0^\infty \frac{-2\alpha x\sin(3x)}{(x^2+\alpha^2)^2}\,dx = -2\alpha\int_0^\infty \frac{x\sin(3x)}{(x^2+\alpha^2)^2}\,dx \qquad \Rightarrow \qquad \int_0^\infty \frac{x\sin(3x)}{(x^2+\alpha^2)^2}\,dx = \frac{m\pi}{4\alpha}e^{-m\alpha}$$

Thus,

$$\int_0^\infty \frac{x\sin(3x)}{(x^2+4)^2}dx = \frac{3\pi}{4(2)}e^{-3(2)} = \frac{3\pi}{8e^6}$$

□

Example: Consider the following integral,

$$I(\alpha) = \int_0^1 \frac{\sin(\alpha x^3)}{\sqrt{1-x^2}}dx$$

Compute $I'(\alpha)$.

Solution:

$$I'(\alpha) = \frac{d}{d\alpha}\int_0^1 \frac{\sin(\alpha x^3)}{\sqrt{1-x^2}}dx = \int_0^1 \frac{\partial}{\partial\alpha}\left[\frac{\sin(\alpha x^3)}{\sqrt{1-x^2}}\right]dx = \int_0^1 \frac{x^3\cos(\alpha x^3)}{\sqrt{1-x^2}}dx$$

□

The following will introduce you to an integration technique known as **Feynman Integration**. The technique will use the theorem stated above regarding differentiating under the integral sign, of an integral which has two variables.

Suppose we wanted to compute the integral

$$\int_0^1 \frac{x^5-1}{\ln x}dx$$

What we can do is replace one of the constants of the integrand with a parameter and define a function based on that parameter. In this case, we will define

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}dx$$

so that what we are trying to find is $I(5)$.

Step 1. Compute the derivative of $I(\alpha)$, as follows

$$I'(\alpha) = \frac{d}{d\alpha}\int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx = \int_0^1 \frac{\partial}{\partial\alpha}\left[\frac{x^\alpha - 1}{\ln x}\right]dx$$

(Notice that the differentiation is with respect to the variable α, so while differentiating you will treat x as a constant. Also, notice that the expression x^α is an exponential term (not a polynomial term) in the variable α). Thus,

$$I'(\alpha) = \int_0^1 \frac{x^\alpha \ln x}{\ln x}\,dx = \int_0^1 x^\alpha\,dx$$

Step 2. Then, compute the integral of the expression involving x and α, treating x as the variable and treating α as a constant.

$$\int_0^1 x^\alpha\,dx = \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_{x=0}^{1} = \frac{1}{\alpha+1}$$

Step 3. You now have $I'(\alpha)$ equal to an expression in α. Take the antiderivative with respect to α, to find $I(\alpha)$. Finally, compute $I(5)$.

$$I'(\alpha) = \frac{1}{\alpha+1} \qquad \Rightarrow \qquad I(\alpha) = \int \frac{1}{\alpha+1}\,d\alpha = \ln|\alpha+1| \qquad \Rightarrow \qquad I(5) = \ln(6)$$

3. WEIERSTRASS SUBSTITUTION

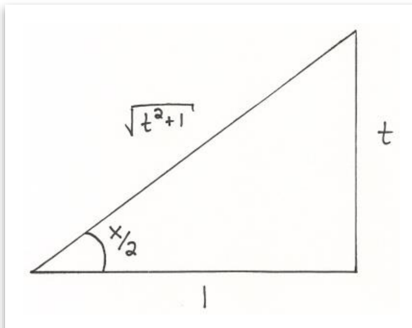
In what follows we will see how to convert an integral involving trigonometric functions into an integral of a rational function. We can then use techniques on this new integral, such as a partial fraction decomposition.

For the substitution, we will use the following change of variables

$$t = \tan\left(\frac{x}{2}\right)$$

From this, we can derive rational expressions in t for the remaining trigonometric functions. This should make sense why we have to do this, since if we are going to replace every expression in x to an expression in t, then we will have to be able to substitute for each trigonometric function.

To find the expressions for the remaining trigonometric functions, we will set up a reference triangle based on the definition of t given above.



From this reference triangle, we can find the sine and cosine of the angle $\frac{x}{2}$,

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{t^2+1}} \qquad \text{and} \qquad \cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}}$$

We want the sine and cosine of the angle x, so the double angle identity gives

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \qquad \Rightarrow \qquad \cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2$$

$$= \frac{1-t^2}{1+t^2}$$

Similarly, we can find the following,

$$\tan x = \frac{2t}{1-t^2} \qquad \sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

$$\csc x = \frac{1+t^2}{2t} \qquad \sec x = \frac{1+t^2}{1-t^2} \qquad \cot x = \frac{1-t^2}{2t}$$

$$dx = \frac{2dt}{1+t^2}$$

Example: Compute the following integral,

$$\int \frac{1}{\sin x + \tan x}\, dx$$

Solution: To compute the integral using a Weierstrass substitution, you will have to substitute for the sine function, the tangent function and the differential. Thus,

$$\int \frac{1}{\sin x + \tan x}\, dx = \int \frac{1}{\left(\frac{2t}{1+t^2} + \frac{2t}{1-t^2}\right)} \cdot \frac{2dt}{1+t^2} = \int \frac{2}{\left(2t + \frac{2t(1+t^2)}{1-t^2}\right)} dt$$

$$= \int \frac{2}{\left(\frac{2t(1-t^2)}{1-t^2} + \frac{2t(1+t^2)}{1-t^2}\right)} dt$$

$$= \int \frac{2}{\left(\frac{2t - 2t^3 + 2t + 2t^3}{1-t^2}\right)} dt = \int \frac{2}{\left(\frac{4t}{1-t^2}\right)} dt = \int \frac{1-t^2}{2t} dt = \int \frac{1}{2t} - \frac{t}{2} dt =$$

$$\frac{1}{2}\ln|t| - \frac{t^2}{4} + c = \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}\right)\right| - \frac{1}{4}\tan^2\left(\frac{x}{2}\right) + c$$

□

Exercises

1. The Even-Odd Decomposition of a Function and Integrals over a Symmetric Interval about the Origin

1. Find the even-odd decomposition of the function. Also, sketch the graph of each function along with its even and odd part. Use a calculator to obtain the graphs.

$a)\ f(x) = x^3 + 2x^2$

$b)\ f(x) = \dfrac{1+\sqrt{x}}{1+x}$

$c)\ f(x) = |x-3|$

$d)\ f(x) = \sqrt[3]{x+1}$

2. If $f(x)$ is an odd function, then prove that the following is true for any value of a,

$$\int_{-a}^{a} f(x)\,dx = 0$$

3. Compute the integral of the polynomial, by only computing the integral of the even part.

$a)\ \displaystyle\int_{-1}^{1} 2x^3 + x^2 + 12x - 1dx$

$b)\ \displaystyle\int_{-2}^{2} -x^5 + 27x^3 + 12x + 1dx$

$c)\ \displaystyle\int_{-3}^{3} -18x^5 + 3x^2 - \pi x + 2dx$

$d)\ \displaystyle\int_{-7}^{7} 3x^{11} - 7x^9 + 3x^7 + 2x^5 + x^3 + 12x + 4dx$

$e)\ \displaystyle\int_{-a}^{a}\left(1 + \sum_{k=1}^{m} x^{2k-1}\right)dx$

4. Compute the following integral by replacing the integrand with its even part.

$a)\ \int_{-3}^{3} \frac{1}{2^x+1}\,dx$

$b)\ \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{(x^2+1)(4^x+1)}\,dx$

$c)\ \int_{-4}^{4} \frac{|x|}{\left(\sqrt{2}\right)^x+1}\,dx$

$d)\ \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{\sin(x)+1}\,dx$

$e)\ \int_{-1}^{1} \frac{x\sin x}{4^x+1}\,dx$

$f)\ \int_{-1}^{1} \frac{\sqrt{1-x^2}}{7^x+1}\,dx$

$g)\ \int_{-1}^{1} \frac{\cos(x)}{e^{(1/x)}+1}\,dx$

$h)\ \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x(2-\sec^2 x)}{\tan x+1}\,dx$

2. FEYNMAN INTEGRATION

5. Use the worked out example in the text to compute the following integrals. That is, use manipulation of the integral.

$a)\ \int_{0}^{1} \frac{x^4-1}{\log_{10} x}\,dx$

$b)\ \int_{0}^{1} \frac{\sqrt{x}-1}{\log_3 x}\,dx$

$c)\ \int_{0}^{1} \frac{x^3+\sqrt[3]{x}-2}{\log_5 x}\,dx$

$d)\ \int_{0}^{1} \frac{x^5+x^2+\sqrt{x}-3}{\log_2(x^4)}\,dx$

6. Compute the stated partial derivatives.

$a)\ \frac{\partial}{\partial x}[x^3+xy^3]$

$b)\ \frac{\partial}{\partial y}[x^3+xy^3]$

$c)\ \frac{\partial}{\partial x}[x^2e^y+ye^{xy}]$

$d)\ \frac{\partial}{\partial t}[-xt+xe^t]$

$e)\ \frac{\partial}{\partial z}[-xt+xe^t]$

$f)\ \frac{\partial}{\partial \alpha}\left[e^{-\alpha x^2}\right]$

$g)\ \frac{\partial}{\partial \alpha}\left[\sqrt{x^2+\alpha^2}\right]$

$h)\ \frac{\partial}{\partial t}\left[\frac{xt}{-x^2+2x-\sin x}\right]$

$i)\ \frac{\partial}{\partial x}[2x^4e^y+xy^x]$

7. We can compute the following integral by finding an antiderivative,

$$\int_{0}^{\infty} \frac{1}{x^2+\alpha^2}\,dx = \frac{\pi}{2\alpha}$$

Use this integral value, let α be a parameter and use differentiation under the integral to find the following integral (make sure to differentiate both sides of the above equation).

$a)\ \displaystyle\int_0^\infty \frac{1}{(x^2+\alpha^2)^2}\,dx$ $\qquad b)\ \displaystyle\int_0^\infty \frac{1}{(x^2+\alpha^2)^3}\,dx$

$c)$ Use your results from parts a and b to compute the following integrals

$$\int_0^\infty \frac{1}{(x^2+9)^2}\,dx \qquad \int_0^\infty \frac{1}{(x^2+16)^2}\,dx \qquad \int_0^\infty \frac{1}{(x^2+4)^3}\,dx \qquad \int_0^\infty \frac{1}{(x^2+5)^3}\,dx$$

8. Consider the following definite integral,

$$\int_0^\infty \frac{\sin(mx)}{x(x^2+\alpha^2)}\,dx = \frac{\pi}{2\alpha^2}\left(1-e^{-m\alpha}\right)$$

Treat α as a parameter and differentiate with respect to α. Then determine the value of the following integral

$$\int_0^\infty \frac{\sin(3x)}{x(x^2+9)^2}\,dx$$

9. Use Feynman integration to compute the following integral,

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

That is, use differentiation under the integral on the following

$$I(\alpha) = \int_0^\infty \frac{\sin(x)}{x} e^{-\alpha x}\,dx$$

then find $I(\alpha)$ and evaluate $I(0)$.

10. Show that the expression $\frac{\arctan(\alpha\tan(x))}{\tan(x)}$ reduces to $x\cot(x)$, when $\alpha = 1$. Then use Feynman integration to compute,

$$\int_0^{\frac{\pi}{2}} x\cot(x)\,dx$$

That is, use differentiation under the integral on the following

$$I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\arctan(\alpha\tan(x))}{\tan(x)}\,dx$$

then find $I(\alpha)$ and evaluate $I(1)$.

11. Use differentiation under the integral to show that the following is true

$$\int_0^{\infty} e^{-\alpha x} \cdot \frac{\cos x - 1}{x}\,dx = \ln\left|\frac{t}{\sqrt{\alpha^2+1}}\right|$$

12. Compute the following using differentiation under the integral sign,

$$I(\alpha) = \int_0^{\infty} \frac{e^{-x} - e^{-\alpha x}}{x}\,dx$$

Then, determine the values of the following

$a)\ \int_0^{\infty} \frac{e^{-x} - e^{-4x}}{x}\,dx$ $\qquad b)\ \int_0^{\infty} \frac{e^{-x} - e^{-e^5 x}}{x}\,dx$ $\qquad c)\ \int_0^{\infty} \frac{e^{-x} - e^{-\sqrt{8}x}}{x}\,dx$ $\qquad d)\ \int_0^{\infty} \frac{e^{-x} - 2^{-3x}}{x}\,dx$

3. WEIERSTRASS SUBSTITUTION

13. Compute the following integral using a Weierstrass substitution. Show that the result reduces to an expression of the form, $-\cos x + C$.

$$\int \sin x \, dx$$

14. Compute the following integral using a partial fraction expansion,

$$\int \frac{1}{x^2-1} \, dx$$

(You will use this antiderivative at some point in the next exercise)

15. Compute the trigonometric integral using a Weierstrass substitution.

$a) \int \frac{1}{1+\cos x} dx$

$b) \int \frac{1}{1+\sin x} dx$

$c) \int \frac{1}{1+\sec x} dx$

$d) \int \frac{1}{2\sin x - \cos x} dx$

$e) \int \frac{1}{1-\sin x + \cos x} dx$

$f) \int \frac{2}{4+5\sin x} dx$

Challenge Problems

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1. (Calc 1) Compute the following limit. Be sure to show all of your work

$$\lim_{x\to 0^+} \frac{\sqrt{4-4\cos^2(2x)}}{\sin(\arcsin(3x))}$$

2. (Calc 1) Compute the following limit

$$\lim_{x\to\infty} \frac{1}{4}\ln(e^5x^2+1) - \frac{1}{2}\ln(x+4)$$

3. (Calc 3) Does the sphere $x^2 + y^2 + z^2 + 6x + 2y - 1 = 0$ intesect the plane $2x - y + 4z = 1$? Why or why not?

4. (Calc 3) Can we compute the integral $\int_0^1 e^{-x^2}\,dx$ using the technique introduced in class for $\int_{-\infty}^{\infty} e^{-x^2}\,dx$? If so, do it. If not, show why we can't.

5. (Precalc) Suppose $f(x)$ is a quadratic function and that $(-2,4)$ is the vertex of $f(x+3)+1$. Use this information to determine the vertex of $-\frac{1}{2}f(x-4)+5$.

6. (Precalc) Let $f(x) = \begin{cases} x-2, & x \leq 0 \\ \frac{1}{3}x + \frac{2}{3}, & x \geq 1 \end{cases}$

$a)$ Sketch the graph of $f(x)$.

$b)$ Explain why $f(x)$ has an inverse function and graph the inverse function.

$c)$ Find the expression (which will be a piecewise function) for the inverse function.

7. (Precalc) Find the domain of $(f \circ f \circ f)(x)$, where $f(x) = \frac{2x}{x-1}$.

8. (Calc 2) Compute the integral using the technique of substitution and using the given u

$$\int \ln(x^6)\, x^{3\ln(x)-1}\left[x^{2\ln(x^3)} - 3\right]dx \qquad\qquad u = x^{\ln(x^3)}$$

9. (Calc 2) Compute the integral

$$\int \frac{x + 2 + \sqrt{5}}{x^2 + 4x - 1}\,dx$$

Hint: Complete the square on the denominator, then factor it using the difference of squares.

10. (Calc 2) Compute the following integral. Hint: use the technique of integration by parts twice. The second application will bring back the original integral, which you can then solve for.

$$\int 2^x \sin x \, dx$$

b) Compute the following integral**. Hint:** Use a change of variables, and then you will use your result from the above integral.

$$\int x^{-\frac{1}{2}} \sin(\log_4 x) \, dx$$

11. (Calc 2) Find the formula for the following integral, where n is an arbitrary positive integer.

$$\int \frac{x^n + x^{n-1} + x^{n-2} + \cdots + x^3 + x^2 + x + 1}{x - 1} dx$$

12. (Calc 2) Compute the following integral

$$\int \frac{\ln x + 1}{x((\ln x)^2 - 4)} dx$$

13. (Calc 2) Compute the following integral

$$\int \frac{(\ln x)^3 \sqrt{9 - (\ln x)^2}}{x} dx$$

14. (Calc 2) Compute the following integral

$$\int \frac{e^x (1 + e^{2x})}{\sqrt{1 - e^{2x}}} dx$$

15. (Calc 2) Find the limit of the following sequence

$$a_n = \sum_{k=1}^{n} \frac{(-1)^{k+1} \left(\sqrt{e} - 1\right)^k}{k}$$

16. (Calc 2) Determine the exact value of the following

$$\ln\left[\left(\sum_{k=1}^{\infty}\frac{\left(\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n+1}\right)^{k}}{k!}\right)^{2}\right]$$

As Brian Robertson points out in his thought provoking article in this issue, mathematics is a collaborative effort, and nothing happens without a supportive and helpful environment.

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